MODULI OF SURFACES WITH AN ANTI-CANONICAL CYCLE

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Abstract. We prove a global Torelli theorem for pairs (Y, D) where Y is a smooth projective rational surface and $D \in |-K_Y|$ is a cycle of rational curves, as conjectured by Friedman in 1984. In addition, we construct natural universal families for such pairs.

Contents

1. Introduction

We work throughout over the field $k = \mathbb{C}$.

Definition 1.1. A *Looijenga pair* (Y, D) is a smooth projective surface Y together with a singular nodal curve $D \in |-K_Y|$. Note $p_a(D) = 1$ by adjunction, so D is either an irreducible genus one curve with a single node, or a cycle of smooth rational curves. We fix an *orientation* of the cycle D, that is, a choice of generator of $H_1(D, \mathbb{Z}) \cong \mathbb{Z}$, and an ordering $D = D_1 + \cdots + D_n$ of the irreducible components of D compatible with the orientation.

By the birational classification of surfaces, Y in Definition 1.1 is necessarily rational.

Looijenga pairs were introduced in [L81] as natural log analogs of K3 surfaces. Looijenga studied the cases $n \leq 5$ in detail. Here we consider moduli of Looijenga pairs with no restriction on n. We prove the global Torelli Theorem, conjectured by Friedman in [F84], see Theorem 1.9. We construct natural universal families, see §3, and we identify the hyperplane arrangement for a natural root system in $D^{\perp} \subset \operatorname{Pic}(Y)$ with the Mori fan of the total space of a universal family, see Theorem 4.14.

The motivation for studying Looijenga pairs comes from several directions. Our initial interest arose from the construction of [GHKI]. There we construct a mirror family to any Looijenga pair (Y, D). If the intersection matrix of the components of D is not negative semi-definite, then our construction yields an algebraic family. We call this the *positive* case. In the sequel [GHKII] to that work, we will apply the Torelli theorem to show that in the positive case the mirror family is the universal family of positive Looijenga pairs constructed here, see Theorem 4.25. This has a striking consequence: our construction of the mirror family endows the fibres with a canonical basis of functions. We call elements of this basis theta functions, as a related construction yields theta functions on abelian varieties. Realizing this as the universal family now endows each affine surface $U = Y \setminus D$ in the family with canonical theta functions. Though these include some of the most classical objects in geometry, e.g., (Y, D) could be a cubic surface with a triangle of lines, in which case U is what Cayley called an affine cubic surface, we do not believe this canonical basis has been previously observed, or even conjectured.

A second application of the universal families will be given in [GHKIII], where we show that Looijenga pairs are closely related to rank two cluster varieties, and realize the Fock-Goncharov fibration of the cluster $\mathcal{X}$-variety (in the rank two case) as a natural quotient of our universal families. (See [FG], [FZ] for the definitions of cluster varieties.) In any event, Looijenga pairs appear in a number of other settings, such as the study of degenerations of $K3$ surfaces: the central fibres for maximal degenerations, type III in Kulikov's classification, are normal crossing unions of such pairs.

Looijenga pairs have an elementary construction:

Definition 1.2. Let $(\bar{Y}, \bar{D})$ be a smooth projective toric surface, where $\bar{D} := \bar{Y} \setminus \mathbb{G}_m^2$ is the toric boundary, i.e., the union of toric divisors of Y. Let $\pi : Y \to \bar{Y}$ be the blowup at some number of smooth points (with infinitely near points allowed) of $\bar{D}$. Let $D \subset \bar{Y}$ be the strict transform of D. Then (Y, D) is a Looijenga pair, and we call $\pi : Y \to \bar{Y}$ a *toric model* for (Y, D).

Essentially all Looijenga pairs arise in this way (i.e., have a toric model). Indeed, define a *simple toric blowup* $(Y', D') \to (Y, D)$ to be the blowup at a node of D, with D' the reduced inverse image of D. A *toric blowup* is a composition of simple toric blowups. Note (Y', D') is again a Looijenga pair, and the log Calabi-Yau is the same, i.e., $Y' \setminus D' = Y \setminus D$. We then have (see [GHKI], Prop. 1.19) the easy fact:

Lemma 1.3. *Given a Looijenga pair (Y, D) there is a toric blowup (Y', D') such that (Y', D') has a toric model.*

For any question we consider, passing to a toric blowup will be at most a notational inconvenience.

Despite the elementary construction, the geometry of Looijenga pairs is surprisingly complex. For example, a given Looijenga pair can have many toric models; we prove that toric models for Y, with fixed $(\bar{Y}, \bar{D})$, are a torsor for a (sometimes infinite) group Adm_Y, closely related to a natural Weyl group, see Theorem 1.13.

To give a precise statement of our results, we first give a number of basic definitions.

Definition 1.4. Let (Y, D) be a Looijenga pair.

(1) A curve $C \subset Y$ is *interior* if no irreducible component of C is contained in D.
(2) An *internal* (-2)*-curve* means a smooth rational curve of self-intersection -2 disjoint from D.
(3) (Y, D) is *generic* if it has no internal (-2)-curves.

Any Looijenga pair is deformation equivalent to a generic pair, see Lemma 3.5. Note by adjunction that any smooth connected interior rational curve is either a (-1)-curve meeting D transversely at a single smooth point, or an internal (-2)-curve.

Note that if (Y, D) is generic and $\pi : Y \to \bar{Y}$ is a toric model, then the blown up points are necessarily distinct (as opposed to infinitely near).

Definition 1.5. An *exceptional configuration* for generic (Y, D) means an ordered collection $E_{ij} \in \mathrm{Pic}(Y)$ of classes of exceptional divisors for a toric model. This is an ordered collection of disjoint internal (-1)-curves. If (Y, D) is not necessarily generic, then by a *limiting configuration* in $\mathrm{Pic}(Y)$ we mean the parallel transport (for the Gauss-Manin connection in a family of Looijenga pairs) of an exceptional configuration on a generic pair.

More generally, we extend the notion of exceptional or limiting configuration to mean the data of a toric blowup $(Y', D') \to (Y, D)$ plus an exceptional or limiting configuration on (Y', D').

We say that two exceptional configurations $\{E_{ij}\}, \{F_{ij}\}$ in $\mathrm{Pic}(Y)$ have the same *combinatorial type* if the associated toric models are blowups of the same toric variety, and if the number of points blown up along a given boundary divisor is the same in either model.

For generic pairs, limiting and exceptional are the same, see Theorem 3.9.

We next consider the notion of periods of Looijenga pairs. We first note (see Lemma 2.1) that the orientation of D determines a canonical identification $\mathbb{G}_m = \mathrm{Pic}^0(D)$, where the latter is the connected component of the identity of $\mathrm{Pic}(D)$.

Definition 1.6. Let

$$D^{\perp} := \{\alpha \in \operatorname{Pic}(Y) \,|\, \alpha \cdot [D_i] = 0 \text{ for all } i\}.$$

Restriction of line bundles determines a canonical homomorphism

$$\phi_Y : D^{\perp} \to \operatorname{Pic}^0(D) = \mathbb{G}_m, \quad L \mapsto L|_D. \tag{1.1}$$

$\phi_Y \in T_{D^{\perp}} := \operatorname{Hom}(D^{\perp}, \mathbb{G}_m)$ is called the *period point* of Y.

Note $Y \setminus D$ comes with a canonical (up to scaling) nowhere-vanishing 2-form, ω. One can show that ϕ_Y is equivalent to the data of periods of ω over cycles in $H_2(Y \setminus D, \mathbb{Z})$, see [F84]. This motivates the term "period."

As well as the notion of periods, we also need the following additional notions to state the Torelli theorem.

Definition 1.7. Let (Y, D) be a Looijenga pair.

(1) The *roots* $\Phi \subset \operatorname{Pic}(Y)$ are those classes in $D^{\perp} \subset \operatorname{Pic}(Y)$ with square -2 and which are realized by an internal (-2)-curve C on a deformation equivalent pair (Y', D'). More precisely, there is a family $(\mathcal{Y}, \mathcal{D})/S$, a path $\gamma\colon [0,1] \to S$, and identifications $(Y, D) = (\mathcal{Y}_{\gamma(0)}, \mathcal{D}_{\gamma(0)})$, $(Y', D') = (\mathcal{Y}_{\gamma(1)}, \mathcal{D}_{\gamma(1)})$, such that the isomorphism
$$H^2(Y', \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$
induced by parallel transport along γ sends $[C]$ to α.

(2) Let $\Delta_Y \subset \operatorname{Pic}(Y)$ be the set of classes of internal (-2)-curves.

(3) Let $\Phi_Y \subset \Phi \subset \operatorname{Pic}(Y)$ be the subset of roots, α, with $\phi_Y(\alpha) = 1$. Observe $\Delta_Y \subset \Phi_Y \subset \Phi$.

(4) Let $W \subset \operatorname{Aut}(\operatorname{Pic}(Y))$ be the subgroup generated by the reflections
$$s_\alpha : \operatorname{Pic}(Y) \to \operatorname{Pic}(Y), \quad \beta \mapsto \beta + \langle \alpha, \beta \rangle \alpha$$
for $\alpha \in \Phi$. Let $W_Y \subset W$ be the subgroup generated by s_α with $\alpha \in \Delta_Y$.

It is clear from the definitions that Φ is invariant under parallel transport, and Δ_Y, Φ_Y, Φ are all invariant under $\operatorname{Aut}(Y, D)$. Further, the sets Φ, Φ_Y, W, W_Y are easily seen to be invariant under toric blowup. Indeed, let $\tau : (Y', D') \to (Y, D)$ be a blow-up of a node of D. Then under pull-back τ^* of divisors, $D^{\perp}$ is isomorphic to $(D')^{\perp}$ as lattices with an intersection form.

We will show that $\Phi_Y = W_Y \cdot \Delta_Y$, see Lemma 4.10.

When $n \leq 5$, Φ contains a natural *root basis*, which is central to much of Looijenga's analysis. No such basis exists in general.

Definition 1.8. Let (Y, D) be a Looijenga pair.

(1) The cone $\{x \in \mathrm{Pic}(Y)_{\mathbb{R}} \mid x^2 > 0\}$ has two connected components. Let C^+ be the connected component containing all the ample classes.
(2) For a given ample H let $\mathcal{M} \subset \mathrm{Pic}(Y)$ be the collection of classes E with $E^2 = K_Y \cdot E = -1$, and $E \cdot H > 0$. Note $\mathcal{M}$ is independent of H, see Lemma 2.10. Let $C^{++} \subset C^+$ be the subcone defined by the inequalities $x \cdot E \geq 0$ for all $E \in \mathcal{M}$.
(3) Let $C^{++}_D \subset C^{++}$ be the subcone where additionally $x \cdot [D_i] \geq 0$ for all i.
(4) A Y-*Weyl chamber* is a connected component of the complement in C^{++} to the union of hyperplanes $\alpha^\perp \cap C^{++}$, $\alpha \in W_Y \cdot \Delta_Y$. A Y-Weyl chamber in C^{++}_D is the intersection of a Y-Weyl chamber with C^{++}_D.

By Lemma 2.10, C^+, C^{++}, C^{++}_D and $\mathcal{M}$ are all independent of deformation of Looijenga pairs (i.e., preserved by parallel transport).

Our main result is then:

Theorem 1.9. *(Torelli Theorem) Let $(Y_1, D), (Y_2, D)$ be Looijenga pairs and let*

$$\mu\colon \mathrm{Pic}(Y_1) \to \mathrm{Pic}(Y_2)$$

be an isomorphism of lattices preserving the intersection pairing.

Global Torelli: *$\mu = f^*$ for an isomorphism of pairs $f : (Y_2, D) \to (Y_1, D)$ iff all the following hold:*

(1) *$\mu([D_i]) = [D_i]$ for all i.*
(2) *$\mu(C^{++}) = C^{++}$.*
(3) *$\mu(\Delta_{Y_1}) = \Delta_{Y_2}$.*
(4) *$\phi_{Y_2} \circ \mu = \phi_{Y_1}$.*

If f exists, the possibilities are a torsor for $\mathrm{Hom}(N', \mathbb{G}_m)$ for N' the cokernel of the map

$$\mathrm{Pic}(Y) \to \mathbb{Z}^n, \quad L \mapsto (L \cdot D_i)_{1 \leq i \leq n}.$$

Weak Torelli: *There is an element g in the Weyl group W_{Y_1} such that $\mu \circ g = f^*$ for an isomorphism of pairs $f : (Y_2, D) \to (Y_1, D)$ iff μ satisfies conditions (1),(2),(4) and $\mu(\Phi) = \Phi$. If g exists, it is unique.*

The global Torelli theorem is proved in §2, the weak in §4.

Remark 1.10. In a preliminary version of this note we claimed the Torelli theorem with (2) replaced by the conditions $\mu(C^+) = C^+$ and $\mu(\Phi) = \Phi$. R. Friedman showed us counterexamples to this statement [F12]. We note the weaker condition $\mu(C^+) = C^+$ is sufficient if D supports a divisor of positive square, or if $\mu(H)$ is ample for some ample H, as either condition is easily seen to imply $\mathcal{M}$, and thus C^{++}, is preserved.

In [F12] Friedman gives various sufficient conditions under which (2) may be replaced by the conditions $\mu(C^+) = C^+$ and $\mu(\Phi) = \Phi$ (all have the flavor of guaranteeing that Φ is sufficiently big).

The proof of the global Torelli theorem is carried out in §2. The key point there is the notion of a marked Looijenga pair and periods for marked Looijenga pairs.

Definition 1.11. Let (Y, D) be a Looijenga pair.

(1) A *marking of* D is a choice of points $p_i \in D_i^o$ for each i, where D_i^o denotes the intersection of D_i with the smooth locus of D. This is equivalent to the choice of an isomorphism $i : D^{\mathrm{can}} \to D$ of D with a fixed cycle of rational curves D^{can}. The possible markings of D are a torsor for $\mathrm{Aut}^0(D) = \mathbb{G}_m^n$, the connected component of the identity of $\mathrm{Aut}(D)$.

(2) Fix (Y_0, D) generic. A marking of $\mathrm{Pic}(Y)$ means an isomorphism of lattices $\mu : \mathrm{Pic}(Y_0) \to \mathrm{Pic}(Y)$ preserving the boundary classes and limiting exceptional configurations of (any given) fixed combinatorial type.

(3) Markings p_i, μ determine a *marked period point*:

$$\phi_{((Y,D),p_i,\mu)} \in T_{Y_0} := \mathrm{Hom}(\mathrm{Pic}(Y_0), \mathbb{G}_m)$$

by

$$\phi(L) := (\mu(L)|_D)^{-1} \otimes \mathcal{O}_D\Big(\sum (L \cdot D_i) p_i\Big) \in \mathrm{Pic}^0(D) = \mathbb{G}_m. \tag{1.2}$$

The global Torelli theorem is proved by first showing that given a toric model for (Y, D), the marked period point determines the location of the blowups, and hence determines Y: this is essentially the content of Proposition 2.7. A bit more work leads to the global Torelli theorem.

For the weak Torelli theorem, there is much more to understand. Our analysis also leads to a number of other results.

The first tool we use is the construction of universal families. Given a generic Looijenga pair (Y_0, D) with a toric model $\pi : Y_0 \to \bar{Y}$ with exceptional configuration $\{E_{ij}\}$, we can construct a *universal family of marked Looijenga pairs* over $T_{Y_0} = \mathrm{Hom}(\mathrm{Pic}(Y_0), \mathbb{G}_m)$. This is obtained by blowing up certain collections of sections q_{ij} of $\bar{Y} \times T_{Y_0} \to T_{Y_0}$ with q_{ij} having image in $D_i^o \times T_{Y_0}$. After appropriately marking the boundary, we obtain a family of marked Looijenga pairs with a fixed isomorphism between $\mathrm{Pic}(Y_0)$ and the Picard group of each fibre of this family. We denote this family by $\mathcal{Y}_{\{E_{ij}\}} \to T_{Y_0}$. It has the feature that the marked period point of the fibre over $\phi \in T_{Y_0}$ is ϕ.

There are in general an infinite number of such universal families containing any given Looijenga pair (Y_0, D). First there are a finite number of choices for the ordering of

the blow-up. Second, there may be an infinite number of different sets of exceptional divisors, even of the same combinatorial type, giving rise to an infinite number of families. We will show that any two universal families are in fact canonically birational, with the birational map an isomorphism in codimension one.

To get a handle on the possibilities for such families, we define:

Definition 1.12. An element μ of $\mathrm{Aut}(\mathrm{Pic}(Y))$ is *admissible* if it preserves the intersection pairing, the classes of the D_i, and takes any limiting configuration on any toric blowup of Y to a limiting configuration. We denote by $\mathrm{Adm}_Y \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ the subgroup of admissible automorphisms.

It is clear that Adm_Y is invariant under toric blowup.

We then have the following characterizations of admissible automorphisms. (1-2) are Theorem 3.9, while (3) is Theorem 4.17.

Theorem 1.13. *Let (Y, D) be a Looijenga pair.*

(1) *Let $\{F_{ij}\}, \{E_{ij}\} \subset \mathrm{Pic}(Y')$ for a toric blowup $Y' \to Y$ be two limiting configurations of the same combinatorial type. There is a unique $g \in \mathrm{Aut}(\mathrm{Pic}(Y))$ preserving the boundary classes with $g(F_{ij}) = E_{ij}$. Furthermore, $g \in \mathrm{Adm}_Y$, and in this way Adm_Y acts simply transitively on limiting configurations in $\mathrm{Pic}(Y')$ of a fixed combinatorial type.*
(2) $\mathrm{Adm}_Y \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ *is the subgroup of monodromy transformations given by parallel transport in families as in Definition 1.7 (1), with $(Y', D') = (Y, D)$.*
(3) Adm_Y *is the subgroup of* $\mathrm{Aut}(\mathrm{Pic}(Y))$ *preserving the intersection pairing, the boundary classes, and C^{++}.*

In particular Adm_Y is preserved by parallel transport. See Theorem 4.16 below for the precise relationship between W and Adm_Y.

We note that each toric model $Y' \to \bar{Y}$ of a toric blowup $Y' \to Y$ determines a copy of an algebraic torus $\mathbb{G}_m^2 = \bar{Y} \setminus D \subset Y \setminus D$, and, by the theorem, the group Adm_Y (which is in many cases infinite) acts simply transitively on these *toric charts* (for fixed Y' and $\bar{Y}$). We find the existence of this multitude of tori in $Y \setminus D$, forming a torsor for a natural group, quite surprising. In the examples Looijenga considers, Adm_Y is a Weyl group associated to a generalized Dynkin diagram. For example a cubic surface together with a triangle of lines has a toric model as a blowup of $\mathbb{P}^2$ at six points, two on each of the components of its standard toric boundary. In this case Looijenga proves Adm_Y is the Weyl group $W(D_4)$.

Remark 1.14. In [GHKIII] we will show that each $U = Y \setminus D$ are related to cluster varieties, and as such are a union of algebraic tori, $\mathbb{G}_m^2$, one for each seed of the cluster.

We believe these cluster tori *are not* the same as the tori parametrized by Adm_Y, and that the Weyl group permutes different (in most cases, infinitely many different) cluster structures on the same U.

A crucial point for us is that the Weyl chambers (Definition 1.8, (4)) have a natural Mori-theoretic meaning in terms of the *Mori fan* of a universal family. Given a universal family $\mathcal{Y} \to T_{Y_0}$ as described above, one considers the set

$$\{\mathrm{Nef}(\mathcal{Y}') \,|\, \mathcal{Y} \dashrightarrow \mathcal{Y}'\}$$

the collection of nef cones (the closure of the ample cones) in $\mathrm{Pic}(\mathcal{Y}) \otimes_{\mathbb{Z}} \mathbb{R}$ of all $\mathcal{Y}'$ isomorphic to $\mathcal{Y}$ off of a codimension two subset. This collection of cones is a fan; for more details see [HK]. In general the Mori fan can be very difficult to control, but for a universal family we can do so. We note that the closure of the moving cone of $\mathcal{Y}$ is the same as the nef cone of a generic Looijenga pair (Y_0, D) in the family, and the Mori fan gives a decomposition of the moving cone. Furthermore, Adm_{Y_0} acts on $\mathrm{Pic}(Y_0) = \mathrm{Pic}(\mathcal{Y})$, and in fact permutes the chambers of the Mori fan. Every universal family has a special fibre over $e \in T_{Y_0}$ (the constant map with value $1 \in \mathbb{G}_m$), and one finds that $\mathrm{Nef}(\mathcal{Y}) = \mathrm{Nef}(\mathcal{Y}_e)$. Using this one can identify the Mori fan for $\mathcal{Y}$ with the set of $\mathcal{Y}_e$-Weyl chambers. This gives sufficient control of the connection between the Weyl group, Adm_Y and the Mori fan to prove many useful results, including weak Torelli.

As well as weak Torelli, we explore in what sense a universal family is universal: the actual moduli space of Looijenga pairs is non-separated, but every Looijenga pair appears in any universal family precisely once. See Theorem 4.20 for a precise statement. In the case that the intersection matrix of D is not negative semi-definite (the *positive case*), we obtain stronger results. In this case the boundary D supports a big and nef divisor which can be used to define a birational morphism contracting all internal (-2)-curves. One then obtains genuinely universal families where all such (-2)-curves are contracted.

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2. The global Torelli Theorem

Lemma 2.1. *Let D be a cycle of n rational curves, with cyclic ordering of the components. This cyclic ordering induces:*

(1) *An identification $\mathrm{Pic}^0(D) = \mathbb{G}_m$, where the former is the group of numerically trivial line bundles.*
(2) *An identification of $\mathrm{Aut}^0(D) = (\mathbb{G}_m)^n$, where the former is the identity component of the automorphism group of D.*

Proof. For (1), the fact that there is an abstract isomorphism $\mathbb{G}_m \cong \mathrm{Pic}^0(D)$ is well-known, and the automorphism group of $\mathbb{G}_m$ as a group is $\{1, -1\}$, so there are only two choices of identification. Here is an explicit construction of an identification determined by the orientation, which will be used throughout. We assume $n \geq 3$, leaving the straightforward modifications for $n = 1, 2$ to the reader. For $L \in \mathrm{Pic}^0(D)$, there is a nowhere-vanishing section $\sigma_i \in \Gamma(L|_{D_i})$. Let $\lambda_i := \sigma_{i+1}(p_{i,i+1})/\sigma_i(p_{i,i+1}) \in \mathbb{G}_m$, where $p_{i,i+1} := D_i \cap D_{i+1}$. Obviously $\lambda(L) := \prod_i \lambda_i$ is independent of the choice of σ_i. The map $L \mapsto \lambda(L)$ gives the canonical isomorphism.

For (2), let (x_i, y_i) be the homogeneous coordinates on D_i with $x_i = 0$ being the point $D_{i-1} \cap D_i$. Then we take the i^{th} copy of $\mathbb{G}_m$ to act on D_i by $(x_i, y_i) \mapsto (x_i, \lambda y_i)$ for $\lambda \in \mathbb{G}_m$. The i^{th} copy of $\mathbb{G}_m$ acts trivially on D_j for $j \neq i$. □

Definition 2.2. Note that the isomorphism class of a toric Looijenga pair $(\bar{Y}, \bar{D})$ is determined by the intersection numbers $\bar{D}_i^2$. Furthermore, a toric model $\pi : (Y, D) \to (\bar{Y}, \bar{D})$ is an iterated blowup at some collection of (not necessarily distinct) points $q_{ij} \in \bar{D}_i^o$ (where $\bar{D}_i^o = \mathbb{G}_m$ is the complement of the nodes of $\bar{D}$ along $\bar{D}_i$). As such, the connected components of the exceptional locus are disjoint unions of chains $E_1 + \cdots + E_r$ of smooth rational curves with self-intersections $-2, -2, \ldots, -1$ (or just a single (-1)-curve), where the length, r, is the number of times we blow up at the same point. This chain supports a unique collection of r reduced connected chains, $C_1, \ldots, C_r$, each of self-intersection -1, ordered by inclusion,

$$C_1 = E_r, C_2 = E_r + E_{r-1}, \quad \ldots \quad C_r = E_r + E_{r-1} + \cdots + E_1.$$

Following Looijenga, we refer to these chains as the *exceptional curves* for this toric model. Each such curve is determined by its class, and they are partially ordered by inclusion. Note if we produce a family of $(\mathcal{Y}, \mathcal{D})$ of Looijenga pairs by varying the points q_{ij} and choosing an order with which to make the iterated blowups, so that in the general fibre we blow up distinct points, then each of these exceptional curves on Y is the limit of a unique smooth exceptional (-1)-curve on the general fibre.

Note (Y, D) together with the classes $\{E_{ij}\}$ of exceptional curves do not determine by themselves the points $q_{ij} \in \bar{Y}$. Indeed, the classes determine a birational contraction $p : (Y, D) \to (W, D)$, and (W, D) is abstractly isomorphic to $(\bar{Y}, D)$, but further data is needed to specify an identification: this is the data of a marking of D. In the next couple of lemmas we show the positions of the q_{ij} are determined by the marked period point. From this the global Torelli result contained in Theorem 1.9 will easily follow.

Lemma 2.3. *Let (Y, D) be a Looijenga pair. For $\alpha \in \operatorname{Aut}^0(D)$ and $L \in \operatorname{Pic}(D)$ let*

$$\psi_\alpha(L) = L^{-1} \otimes \alpha^*(L) \in \operatorname{Pic}^0(D)$$

This gives a homomorphism $\psi : \operatorname{Aut}^0(D) \to \operatorname{Hom}(\operatorname{Pic}(Y), \operatorname{Pic}^0(D))$ via

$$\psi(\alpha)(L) = \psi_\alpha(L|_D).$$

Under the identifications $\operatorname{Aut}^0(D) = \mathbb{G}_m^n$, $\operatorname{Pic}^0(D) = \mathbb{G}_m$ of Lemma 2.1,

$$\psi_{(\lambda_1,\ldots,\lambda_n)}(L) = \prod_i \lambda_i^{\deg L|_{D_i}}$$

for $L \in \operatorname{Pic}(D)$.

Proof. It's enough to compute $\psi_{(1,\ldots,1,\lambda,1,\ldots,1)}(\mathcal{O}_D(q))$ for $q \in D_j^o$, where λ is in the i^{th} spot. Clearly this is $\lambda^{\delta_{ij}}$, as required. □

We write $\operatorname{Aut}(Y, D)$ for the automorphisms of a Looijenga pair (Y, D) preserving each component of D and the orientation of D.

Proposition 2.4. *There is a long exact sequence*

$$\begin{aligned} 1 \to \ker[\operatorname{Aut}(Y, D) \to \operatorname{Aut}(\operatorname{Pic}(Y))] \to \operatorname{Aut}^0(D) & \\ \xrightarrow{\psi} \operatorname{Hom}(\operatorname{Pic}(Y), \operatorname{Pic}^0(D)) \to \operatorname{Hom}(D^\perp, \operatorname{Pic}^0(D)) \to 1 & \end{aligned}$$

where ψ is the map of Lemma 2.3 and the other maps are the canonical restrictions.

Proof. It is easy to see that if $(Y', D') \to (Y, D)$ is a toric blowup, then the result for (Y', D') implies the result for (Y, D), so by Lemma 1.3 we can assume (Y, D) has a toric model $\pi : Y \to \bar{Y}$.

We have the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{Pic}(\bar{Y}) \arrow[r] \arrow[d, "\pi^*"] & \bigoplus_i \mathbb{Z}\bar{D}_i \arrow[r] \arrow[d, "="] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D^{\perp} \arrow[r] \arrow[d] & \mathrm{Pic}(Y) \arrow[r] \arrow[d] & \bigoplus_i \mathbb{Z}D_i \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus \mathbb{Z}E_{ij} \arrow[r, "="] & \bigoplus \mathbb{Z}E_{ij} \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\
 & & 0 & & &
\end{tikzcd}$$

Here N is dual to the character lattice, M, of the structure torus of $\bar{Y}$. The first row is the standard description of $H_2(\bar{Y}, \mathbb{Z})$, identified with $\mathrm{Pic}(\bar{Y})$ by Poincaré duality, with the map from $\mathrm{Pic}(\bar{Y})$ given by $C \mapsto \sum_i (C \cdot \bar{D}_i)\bar{D}_i$. The map to N takes $\bar{D}_i$ to the first lattice point v_i along the ray of the fan corresponding to $\bar{D}_i$. This exact sequence is the dual of the standard exact sequence describing $\mathrm{Pic}(\bar{Y})$, see e.g., [Fu93], §3.4. The E_{ij}'s are the exceptional divisors of π. The map $\mathrm{Pic}(Y) \to \bigoplus_i \mathbb{Z}D_i$ is similarly given by $C \mapsto \sum_i (C \cdot D_i)D_i$.

The kernel of $N \to N'$ is easily seen to be the subgroup $S \subset N$ generated by the rays in the fan for $\bar{Y}$ corresponding to boundary divisors $\bar{D}_i$ along which π is not an isomorphism.

Note that $N = \mathrm{Hom}(N, \bigwedge^2 N)$ via $n \mapsto (n' \mapsto n' \wedge n)$ and the orientation gives a trivialization $\bigwedge^2 N = \mathbb{Z}$, thus an identification $N = M$. Thus

$$\mathrm{Hom}(N/S, \mathbb{G}_m) \subset \mathrm{Hom}(N, \mathbb{G}_m) = \mathrm{Hom}(M, \mathbb{G}_m)$$

is the subgroup of homomorphisms to $\mathbb{G}_m$ whose restriction to S is trivial. Equivalently, these are the automorphisms in $\mathrm{Aut}(\bar{Y}, \bar{D}) = \mathrm{Hom}(M, \mathbb{G}_m)$ fixing pointwise those $\bar{D}_i$ along which π is not an isomorphism. It's easy to see this is identified with

$$\ker\big(\mathrm{Aut}(Y, D) \to \mathrm{Aut}(\mathrm{Pic}(Y))\big).$$

The result follows by applying $\mathrm{Hom}(\cdot, \mathrm{Pic}^0(D))$ to the row of the above commutative diagram describing $\mathrm{Pic}(Y)$. The fact that the middle map coincides with ψ then follows from Lemma 2.3. $\square$

We next show that for a toric Looijenga pair, any possible marked period point can be realised by a particular choice of marking of D.

Lemma 2.5. *Let $(\bar{Y}, D = D_1 + \cdots + D_n)$ be a toric Looijenga pair, including a choice of structure torus $T \subset \bar{Y}$. Let $\bar{\phi} \in \operatorname{Hom}(\operatorname{Pic}(\bar{Y}), \operatorname{Pic}^0(D))$. Then there are points $p_i \in D_i^o \subset \bar{Y}$ such that*

$$\bar{\phi}(L) = (L|_D)^{-1} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D((L \cdot D_i)p_i).$$

Moreover, T acts simply transitively on the possible collections of p_i.

Proof. Start with an arbitrary choice of $p_i \in D_i^o$. The exact sequence of Lemma 2.4 reduces to

$$1 \longrightarrow T \longrightarrow \operatorname{Aut}^0(D) \xrightarrow{\psi} \operatorname{Hom}(\operatorname{Pic}(\bar{Y}), \operatorname{Pic}^0(D)) \longrightarrow 1.$$

Denote the map $L \mapsto (L|_D)^{-1} \otimes \bigotimes_{i=1}^n \mathcal{O}_D((L \cdot D_i)p_i)$ by $\bar{\phi}' \in \operatorname{Hom}(\operatorname{Pic}(\bar{Y}), \operatorname{Pic}^0(D))$. Given any $\alpha \in \operatorname{Aut}^0(D)$, using Lemma 2.3, consider the map

$$\begin{aligned} L \mapsto (L|_D)^{-1} &\otimes \bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)\alpha^{-1}(p_i)\big) \\ &= \bar{\phi}'(L) \otimes \psi_\alpha \left(\bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)p_i\big) \right) \\ &= \bar{\phi}'(L) \otimes \psi(\alpha)(L). \end{aligned}$$

So this map coincides with $\bar{\phi}' \otimes \psi(\alpha)$. Thus by replacing p_i with $\alpha^{-1}(p_i)$ for some suitable choice of α, we obtain $\bar{\phi} = \bar{\phi}'$. Furthermore, the possible choices of p_i are a torsor for the kernel of ψ. □

Lemma 2.6. *Let $(\bar{Y}, D)$ be as in Lemma 2.5. The structure of $\bar{Y}$ as a toric variety gives a canonical identification of D_i^o with $\mathbb{G}_m$. Let $m_i \in D_i^o$ correspond to $-1 \in \mathbb{G}_m$ under this identification. Define*

$$p_i : \operatorname{Aut}^0(D) \to D_i^o, \alpha \mapsto \alpha^{-1}(m_i).$$

(1)
$$\psi(\alpha)(L) = (L|_D)^{-1} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)p_i(\alpha)\big) \in \operatorname{Pic}^0(D)$$
for all $\alpha \in \operatorname{Aut}^0(D)$ (ψ as in Lemma 2.3).

(2) *Noting ψ is surjective, let $\gamma : \operatorname{Hom}(\operatorname{Pic}(\bar{Y}), \operatorname{Pic}^0(D)) \to \operatorname{Aut}^0(D)$ be a section of ψ. Let $\bar{p}_i : \operatorname{Hom}(\operatorname{Pic}(\bar{Y}), \operatorname{Pic}^0(D)) \to D_i^o$ be the composition $p_i \circ \gamma$. Then for each $\bar{\phi} \in \operatorname{Hom}(\operatorname{Pic}(\bar{Y}), \operatorname{Pic}^0(D))$, the points $\bar{p}_i(\bar{\phi})$ satisfy the conclusion of Lemma 2.5 for $\bar{\phi}$.*

Proof. (1) amounts to showing that

$$L|_D = \bigotimes_{i=1}^{n} \mathcal{O}_D((L \cdot D_i)m_i).$$

It's enough to do this for an ample line bundle, so we can assume $(\bar{Y}, L)$ is the polarized toric surface given by a lattice polygon. In that case take the section of L given by a sum of monomials corresponding to all lattice points on the boundary, with coefficients chosen so that the restriction of the section to $D_i \cong \mathbb{P}^1$ takes the form $(x+y)^{L \cdot D_i}$. Its zero scheme is exactly $\sum (L \cdot D_i)m_i$.

For (2), note

$$\bar{\phi}(L) = \psi(\gamma(\bar{\phi}))(L) = (L|_D)^{-1} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_D\big((L \cdot D_i)\bar{p}_i(\bar{\phi})\big)$$

by (1), as desired. □

The following contains most of the ideas needed for global Torelli, showing that the marked period point determines a marked Looijenga pair.

Proposition 2.7. *Let (Y, D) be a Looijenga pair and $\{E_{ij}\} \subset \mathrm{Pic}(Y)$ the classes of exceptional curves for a toric model of type $(\bar{Y}, D)$. Let $\phi \in \mathrm{Hom}(\mathrm{Pic}(Y), \mathrm{Pic}^0(D))$.*

(1) *There is an inclusion $\mathrm{Pic}(\bar{Y}) \subset \mathrm{Pic}(Y)$ given by pullback. Let $\bar{\phi} : \mathrm{Pic}(\bar{Y}) \to \mathrm{Pic}^0(D)$ be the restriction $\phi|_{\mathrm{Pic}(\bar{Y})}$. Let $p_i \in D_i^o \subset \bar{Y}$ be given by $\bar{\phi}$ from Lemma 2.5. There are unique points $q_{ij} \in D_i^o \subset \bar{Y}$ such that*

$$\phi(E_{ij}) = \mathcal{O}_D(q_{ij})^{-1} \otimes \mathcal{O}_D(p_i).$$

Let (Z, D) be the iterated blowup along the collection of points (possibly with repetitions) $q_{ij} \subset D_i^o \subset \bar{Y}$. There is a unique isomorphism $\mu : \mathrm{Pic}(Y) \to \mathrm{Pic}(Z)$ preserving boundary classes, and sending E_{ij} to the class of the corresponding exceptional curve. Under this identification, ϕ is the marked period point of $((Z, D), p_i, \mu)$, as defined in (1.2).

(2) *Suppose there is a marking $r_i \in D_i^o \subset Y$ so that ϕ is the marked period point for $((Y, D), r_i)$. Then μ is induced by a unique isomorphism of Looijenga pairs between (Y, D) and (Z, D) which sends r_i to p_i.*

Proof. (1) is immediate from the construction. So we assume we have the marking $r_i \in D_i^o$ as in (2). By assumption there is a birational map $\pi : Y \to \bar{Y}$ with exceptional curves $\{E_{ij}\}$, and $\pi^* : \mathrm{Pic}(\bar{Y}) \to \mathrm{Pic}(Y)$ is the inclusion of (1). Now by definition of the marked period point, the points $\pi(r_i)$ satisfy the conclusions of Lemma 2.5 for $\bar{\phi}$. Thus by the uniqueness statement in that lemma, we can change π (composing by a translation in the structure torus of $\bar{Y}$) and assume $\pi(r_i) = p_i$. The points $\pi(E_{ij} \cap D_i)$

satisfy the conditions on the q_{ij}, so by uniqueness $\pi(E_{ij} \cap D_i) = q_{ij}$. Thus π is exactly the same iterated blowup as Z, and so clearly (Y, D) and (Z, D), together with the markings of their boundaries, are isomorphic, by an isomorphism inducing μ. This isomorphism is unique by Proposition 2.4. $\square$

Corollary 2.8. *Let $(Y, D), (Y', D)$ be Looijenga pairs (resp. pairs with marked boundary), having toric models of the same combinatorial type. Let ϕ, ϕ' be the period points (resp. the marked period points). Then there is a unique isomorphism of lattices $\mu : \operatorname{Pic}(Y) \to \operatorname{Pic}(Y')$ preserving the boundary classes and the exceptional curves for the toric models. The isomorphism μ is induced by an isomorphism f of Looijenga pairs (resp. pairs with marked boundary) iff $\phi' \circ \mu = \phi$, and in that case the possible f form a torsor for*

$$\ker\big(\operatorname{Aut}(Y, D) \to \operatorname{Aut}(\operatorname{Pic}(Y))\big)$$

(resp. f is unique).

Proof. The marked case is immediate from Proposition 2.7. For the unmarked case, write $\bar{\phi}, \bar{\phi}'$ for the period points defined by (1.1). Choose arbitrary markings of the boundaries of Y, Y', with marked period points ϕ, ϕ'. Now by Proposition 2.4 we can adjust the marking of the boundary of Y so $\phi = \phi'$. The final torsor statement is clear from Proposition 2.4. $\square$

Lemma 2.9. *Let (Y, D) be a Looijenga pair. Let $\mathcal{M} \subset \operatorname{Pic}(Y)$ be the set of classes of smooth (-1)-curves not contained in D.*

1. *Let $C \subset Y$ be an irreducible curve. Either $C^2 \geq 0$ or $[C] \in \operatorname{Pic}(Y)$ is in the convex hull of the union of $\mathcal{M}$, Δ_Y and the classes $[D_i]$ of the n irreducible components of D.*
2. *Let $H \in \operatorname{Pic}(Y)$ be an ample class. Then the closure of the Mori cone of curves $\overline{\operatorname{NE}(Y)}$ is the closure of the union of the convex hull of*

$$C^+ := \{x \in \operatorname{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \mid x^2 > 0, x \cdot H > 0\}$$

together with Δ_Y, $\mathcal{M}$ and $\{[D_i] \mid 1 \leq i \leq n\}$.

Proof. For (1), let $C \subset Y$, $C \not\subset D$, be irreducible. If $C^2 < 0$ then $C \in \Delta_Y \cup \mathcal{M}$ by adjunction.

For (2), note $C^+ \subset \operatorname{NE}(Y)$ by Riemann-Roch and if C is effective with $C^2 \geq 0$, then C is contained in the closure of C^+. The description of the Mori cone then follows from (1) and the fact that $E \in \mathcal{M}$ is effective by Riemann-Roch. $\square$

Lemma 2.10. *Let (Y, D) be a Looijenga pair. Let $E \in \operatorname{Pic}(Y)$ be a class with $E^2 = K_Y \cdot E = -1$. The following are equivalent*

(1) *$E \cdot H > 0$ for some nef and big divisor H.*
(2) *$E \cdot H > 0$ for all ample divisors H*

The cones C^{++} and C_D^{++} defined in Definition 1.8 are invariant under parallel transport for deformations of Looijenga pairs, and under the action of W_Y.

Proof. Obviously (2) implies (1). As in the previous proof, (1) implies E is effective by Riemann-Roch, which gives (2).

Given a family of Looijenga pairs, we can choose an ample divisor, H, on the total space and then compute C^{++} on each fibre using the restriction of H. From this deformation invariance is clear. Finally, suppose $\alpha \subset Y$ is a smooth rational curve of self-intersection -2, and let L be ample. Note $H := L + ((L \cdot \alpha)/2)\alpha$ is nef and big, and $H \cdot C = 0$ for an effective curve C iff C is supported on α. Note H is preserved by reflection in α. Now take $E \in \mathcal{M}$ as in the definition of C^{++}. Then E is effective, and obviously not supported on α. Thus $H \cdot E > 0$, so

$$0 < H \cdot E = s_\alpha(H) \cdot s_\alpha(E) = H \cdot s_\alpha(E).$$

It follows that $s_\alpha(C^{++}) \subset C^{++}$. The analogous statements for C_D^{++} follow. □

Lemma 2.11. *Let (Y, D) be a Looijenga pair and $H \in \mathrm{Pic}(Y)$ an ample class. Then $\mathrm{Nef}(Y) \subset H^2(Y, \mathbb{R})$ is the closure of the subcone of C_D^{++} defined by the inequalities $x \cdot \alpha > 0$ for all $\alpha \in \Delta_Y$.*

Proof. Since $\mathrm{Nef}(Y)$ is the dual cone to $\mathrm{NE}(Y)$, this follows immediately from Lemma 2.9, (2). □

Proof of the global Torelli, Theorem 1.9. If $\mu = f^*$ for an isomorphism f, μ obviously satisfies the conditions, and the possibilities for f are $\ker\big(\mathrm{Aut}(Y, D) \to \mathrm{Aut}(\mathrm{Pic}(Y))\big)$, as in Corollary 2.8. This is identified in the proof of Proposition 2.4 with $\mathrm{Hom}(N', \mathbb{G}_m)$.

Now assuming we have such a μ, we show it is induced by an isomorphism of pairs. We can replace Y_1 by a toric blowup and Y_2 by the corresponding toric blowup, and so by Lemma 1.3 we can assume Y_1 has a toric model. Then $\mu(\mathrm{Nef}(Y_1)) = \mathrm{Nef}(Y_2)$ by Lemma 2.11. Thus the same is true of Mori cones by duality. Of course $\mu(K_{Y_1}) = K_{Y_2}$ since D is anti-canonical. Now observe that the existence of a toric model of a given combinatorial type can be read off from the Mori cone, together with the classes $[D_i]$, and the class K_{Y_i} (it's a question of the existence of a face of the Mori cone with certain combinatorial properties vis a vis K_{Y_i} and the $[D_i]$). Thus Y_2 has a toric model of the same combinatorial type. So we may apply Corollary 2.8. □

We will prove the weak Torelli Theorem after we have studied the birational geometry of universal families, which we turn to next.

3. Universal families and admissible maps

Recall from Definition 1.11 that if $((Z,D),p_i)$ is a Looijenga pair with marked boundary, and $\mu : \mathrm{Pic}(Y) \to \mathrm{Pic}(Z)$ is an isomorphism, the marked period point of $((Z,D),p_i,\mu)$ is a point in

$$T_Y := \mathrm{Hom}(\mathrm{Pic}(Y), \mathrm{Pic}^0(D)).$$

Construction 3.1. *Universal families.* Let (Y,D) be a Looijenga pair, and $\pi : Y \to \bar{Y}$ a toric model, with exceptional divisors $\{E_{ij}\}$ which are disjoint interior (-1)-curves. Varying $\phi \in T_Y$, the construction of Proposition 2.7 produces sections $p_i : T_Y \to T_Y \times D_i^o \subset T_Y \times \bar{Y}$, and then unique sections $q_{ij} : T_Y \to T_Y \times D_i^o$ such that

$$\phi(E_{ij}) = \mathcal{O}_D(q_{ij}(\phi))^{-1} \otimes \mathcal{O}_D(p_i(\phi)) \in \mathrm{Pic}^0(D) = \mathbb{G}_m.$$

Explicitly, take $p_i = \bar{p}_i$ of Lemma 2.6 (this involves choosing the section γ, but see Remark 3.2) and then $q_{ij}(\phi) \in \mathbb{G}_m$ is the point

$$\phi(E_{ij})^{-1} \cdot p_i(\phi) \in D_i^o,$$

where we think of $\mathrm{Pic}^0(D) = \mathbb{G}_m$ as acting on D_i^o using the convention of Lemma 2.1.

Let $\Pi : (\mathcal{Y}_{\{E_{ij}\}}, \mathcal{D}) \to T_Y \times \bar{Y}$ be the iterated blowup along the sections

$$q_{ij} \subset T_Y \times D_i^o \subset T_Y \times \bar{Y}.$$

This comes with a marking $\mu : \mathrm{Pic}(Y) \to \mathrm{Pic}(\mathcal{Y})$ preserving boundary classes, and sending E_{ij} to the corresponding exceptional divisor $\mathcal{E}_{ij}$. This induces a marking of $\mathrm{Pic}(Z)$ for each fibre Z. We call $\lambda : (\mathcal{Y}_{\{E_{ij}\}}, p_i, \mu) \to T_Y$ a *universal family*, a term we will justify shortly.

If $\tau : Y \to Y'$ is a toric blowup, with exceptional divisor E, and Y has a toric model as above, then there is a divisorial contraction $\tilde{\tau} : \mathcal{Y}_{\{E_{ij}\}} \to \tilde{\mathcal{Y}}'_{\{E_{ij}\}}$ which blows down the (-1)-curve $\mu(E)$ in each fibre — this is a family of toric blowups. Observe that identifying $\mathrm{Pic}(Y)$, $\mathrm{Pic}(Y')$ with $A_1(Y)$, $A_1(Y')$ respectively, we have a map $\tau_* : A_1(Y) \to A_1(Y')$, and hence a transpose map $T_{Y'} = \mathrm{Hom}(A_1(Y'), \mathbb{G}_m) \to \mathrm{Hom}(A_1(Y), \mathbb{G}_m) = T_Y$, an inclusion of tori. This identifies $T_{Y'}$ with the elements of T_Y which take the value 1 on exceptional divisors of τ. We define $\lambda' : \mathcal{Y}'_{\{E_{ij}\}} \to T_{Y'}$ to be the restriction of $\tilde{\mathcal{Y}}'_{\{E_{ij}\}}$ to $T_{Y'} \subseteq T_Y$. This inherits markings of the boundary and the Picard group. In this way we have a universal family associated with each configuration of exceptional curves for a toric model of some toric blowup.

Remark 3.2. Note in the construction we made a choice of right inverse $\gamma : T_{\bar{Y}} \to \mathrm{Aut}^0(D)$ of ψ. By Proposition 2.4, any two choices differ by a homomorphism $h : T_{\bar{Y}} \to \mathrm{Aut}(\bar{Y}, D)$. One can check that h together with the action of $\mathrm{Aut}(\bar{Y}, D)$ on $\bar{Y}$ induces a canonical identification of the universal families constructed.

Remark 3.3. There are in general infinitely many universal families of a given combinatorial type. For a given pair (Y, D) with exceptional divisors E_{ij} for a toric model, the above construction gives a finite number of families, as there is a choice of order of blowup. However, there may be an infinite number of sets of exceptional divisors of the same combinatorial type, giving rise to an infinite number of families. We will see that any two are birational, canonically identified by a birational map, see Construction-Theorem 4.1.

By construction:

Lemma 3.4. *For $\phi \in T_Y$, the period point of the fibre $((\mathcal{Y}, \mathcal{D}), p_i, \mu)_\phi$ of a universal family is ϕ.*

Corollary 3.5. *Any Looijenga pair is a deformation of a generic pair.*

Proof. It's easy to reduce to the case when (Y, D) has a toric model. It is clear that any two Looijenga pairs with toric models of the same type are deformation equivalent — there is an obvious family, as in the construction of the universal family, parametrizing the possible iterated blowups of a given toric $(\bar{Y}, D)$. So it is enough to show that there are generic pairs in each universal family. Let $\alpha \in \operatorname{Pic}(Y)$. If, for a fibre (Z, D) of a universal family, $\alpha \in \operatorname{Pic}(Z)$ represents a curve disjoint from D, then by Lemma 3.4, the period (Z, D) in T_Y lies on the hypertorus

$$T_\alpha := \{\phi \in T_Y | \phi(\alpha) = 1\}.$$

There are only countably many such hypertori. $\square$

Definition 3.6. Let (Y_i, D), $i = 1, 2$ be generic Looijenga pairs. An isomorphism $\mu : \operatorname{Pic}(Y_1) \to \operatorname{Pic}(Y_2)$ is called *admissible* if μ preserves the intersection forms, the classes of the boundary divisors, and gives a bijection on exceptional configurations (see Definition 1.5) on any toric blowup. Let $\operatorname{Adm}_Y \subset \operatorname{Aut}(\operatorname{Pic}(Y))$ be the subgroup of admissible automorphisms. Note this is invariant under toric blowup.

We now show that the universal families constructed above are in some sense universal, and that exceptional configurations deform to exceptional configurations generically.

Lemma 3.7. *Let $(\mathcal{Z}, \mathcal{D}) \to S$ be a one-parameter family of Looijenga pairs, $\mathcal{C} \subseteq \mathcal{Z}$ an irreducible effective divisor such that (1) the composition $\mathcal{C} \to S$ is a flat family and (2) the general fibre of $\mathcal{C} \to S$ is an irreducible rational curve meeting $\mathcal{D}$ at one point. Then every fibre of $\mathcal{C} \to S$ meets $\mathcal{D}$ in a single point.*

Proof. The only way this could fail is if there is a $(Z,D) := (\mathcal{Y}_s, \mathcal{D}_s)$ for some $s \in S$ such that $C := \mathcal{C}_s$ has an irreducible component contained in $\mathcal{D}_s$. Since C must be of genus zero, C cannot contain all of D. Thus either there are two distinct components D_i, D_j of D not contained in C but which each intersect C in at least one point, or there is a component D_i intersecting C in at least two points. On the other hand, $\mathcal{D}_i \cap \mathcal{C}$ is a divisor on $\mathcal{D}_i$ and hence dominates S, and the same for $\mathcal{D}_j$. This clearly contradicts the assumption that the general fibre of $\mathcal{C} \to S$ only intersects $\mathcal{D}$ in one point. □

Proposition 3.8. *Let $\lambda : (\mathcal{Z}, \mathcal{D}) \to S$ be a family of Looijenga pairs over an irreducible base S, with markings of the boundary $p_i : S \to \mathcal{D}$. Suppose generic (Y_0, D) occurs as a fibre, and there are classes $\mathcal{E}_{ij} \in \operatorname{Pic}(\mathcal{Z})$ whose restriction to (Y_0, D) are an exceptional configuration for the toric model $\pi : Y_0 \to \bar{Y}$.*

(1) *The classes $\mathcal{E}_{ij}$ and $\mathcal{D}_i$ determine a canonical section $\sigma : \operatorname{Pic}(Y_0) \to \operatorname{Pic}(\mathcal{Z})$ of the restriction map $\operatorname{Pic}(\mathcal{Z}) \to \operatorname{Pic}(Y_0)$. Thus for any fibre (Z, D) of λ over a closed point $s \in S$, we have a marking*

$$\mu : \operatorname{Pic}(Y_0) \to \operatorname{Pic}(Z)$$

given by the composition of σ with the restriction map $\operatorname{Pic}(\mathcal{Z}) \to \operatorname{Pic}(Z)$. In particular, one obtains a marked period mapping

$$\phi := \phi_{((\mathcal{Z},\mathcal{D}),p_i,\mu)} : S \to T_{Y_0} = \operatorname{Hom}(\operatorname{Pic}(Y_0), \operatorname{Pic}^0(D))$$

defined by (1.2).

(2) *Let (Z, D) be a fibre of λ over a closed point $s \in S$. We have*

$$\phi(s)(\mathcal{E}_{ij}|_Z) = \mathcal{O}_D(-q_{ij}) \otimes \mathcal{O}_D(p_i(s))$$

for unique points $q_{ij} \in D_i$. Then there is a toric model $(Z, D) \to (\bar{Y}, D)$ which is the iterated blowup at the points $q_{ij} \in D \subset \bar{Y}$. If (Z, D) is generic, then μ is admissible.

(3) *For each closed point $s \in S$, $((\mathcal{Z}, \mathcal{D}), p_i)_s$ is isomorphic to the corresponding fibre $((\mathcal{Y}, \mathcal{D}), p_i)_{\phi(s)}$ of any universal family.*

(4) *If every fibre is generic, then $((\mathcal{Z}, \mathcal{D}), p_i, \mu)$ is isomorphic to the pull-back by ϕ of any universal family, and the isomorphism is unique. Moreover if $\phi' : S \to T_{Y_0}$ is a map such that the pull-back of a universal family over T_{Y_0} via ϕ' is isomorphic to $((\mathcal{Z}, \mathcal{D}), p_i, \mu)$, then $\phi(s) = \phi'(s)$ for all closed points $s \in S$.*

Proof. (1) is clear since the classes of the boundary curves and exceptional curves of π generate $\operatorname{Pic}(Y_0)$.

To show (2)-(4), we first note that in (4) the isomorphism is unique, if it exists, by Proposition 2.4. The final claim in (4) follows from Lemma 3.4, which implies distinct fibres of a universal family are not isomorphic as marked Looijenga pairs. We now induct on the Picard number of the fibres of λ. If this is the same as the Picard number of $\bar{Y}$ then $(\mathcal{Z}, \mathcal{D})$ is a fibre bundle with fibre $(\bar{Y}, D)$. Then ϕ and p_i satisfy Lemma 2.5 by definition of ϕ, as does the pull-back of any universal family (with its markings) under $\phi : S \to T_{Y_0}$. Now by the simple transitivity statement of Lemma 2.5, $((\mathcal{Z}, \mathcal{D}), p_i)$ is isomorphic to the pull-back.

Now we consider higher Picard number. Let Z be a closed fibre over a point $s \in S$. By Lemma 3.7, the restriction $\mathcal{E}_{ij}|_Z$ is the class of an interior curve, i.e., has no components contained in D. By the intersection numbers it follows that $\mathcal{E}_{ij}|_Z =: F_{ij} = G_{ij} + H_{ij}$, for a connected curve F_{ij}, and where G_{ij} is reduced and irreducible, $G_{ij} \cdot D_k = \delta_{ik}$, and $H_{ij} \cap D = \emptyset$.

Now let $\Delta = e(\sum \mathcal{E}_{ij})$ for $e > 0$. Since Looijenga pairs have smooth versal deformation space [L81, II.2.4], working locally analytically near $s \in S$ we may assume that S is smooth. Then, for e sufficiently small, $K_{\mathcal{Z}} + \mathcal{D} + \Delta$ is log canonical. We run the relative $K_{\mathcal{Z}} + \mathcal{D} + \Delta$-minimal model program over S, and consider the contraction of an extremal ray. Suppose first the contraction is divisorial, $\mathcal{Z} \to \mathcal{Z}'$. Since $K_{\mathcal{Z}} + \mathcal{D}$ is relatively trivial, the exceptional divisor is some $\mathcal{E}_{ij}$; in particular, the contraction is $K_{\mathcal{Z}}$-negative and thus H_{ij} is empty (otherwise H_{ij} is $K_{\mathcal{Z}}$-trivial and contracted). Now by adjunction G_{ij} is a smooth (-1)-curve, so $(\mathcal{Z}', \mathcal{D})$ is again a family of Looijenga pairs, to which the inductive hypothesis applies. We have a factorization $Y_0 \to Y_0' \to \bar{Y}$, with $b : Y_0 \to Y_0'$ the blowdown of E_{ij} (the exceptional divisor corresponding to the divisorial contraction). We also have the blow-down $Z \to Z'$. By the inductive hypothesis, there is a toric model $(Z', D) \to (\bar{Y}, D)$ as in (2), and it is clear that Z is obtained from Z' by blowing up the point q_{ij}.

Pullback $b^* : \mathrm{Pic}(Y_0') \to \mathrm{Pic}(Y_0)$ induces a homomorphism

$$b_* : T_{Y_0} = \mathrm{Hom}(\mathrm{Pic}(Y_0), \mathbb{G}_m) \to T_{Y_0'} := \mathrm{Hom}(\mathrm{Pic}(Y_0'), \mathbb{G}_m).$$

Pick a universal family $\mathcal{Y}' \to T_{Y_0'}$ and pull-back this family to get $\bar{\mathcal{Y}}' \to T_{Y_0}$. The fibre of this family over $\phi(s)$ is isomorphic to the fibre of $\mathcal{Y}'$ over $\phi(s) \circ b_* \in T_{Y_0'}$, and this is isomorphic to Z' by the induction hypothesis. It then follows easily that Z is isomorphic to the fibre of $\mathcal{Y}$ over $\phi(s)$, as $\mathcal{Y}_{\phi(s)}$ is obtained by blowing up $\mathcal{Y}'_{\phi(s)}$ at q_{ij}, by construction of $\mathcal{Y}$. (4) follows similarly.

Next suppose we have a small contraction. Let $E \subset Z$ be an irreducible exceptional curve. Necessarily $E \subseteq \mathcal{E}_{ij}$ for some i, j as E is Δ-negative. So $E \subseteq F_{ij}$. If $E = G_{ij}$, then G_{ij} is contractible and so by adjunction a (-1)-curve. But then it deforms to the

general fibre, so the contraction cannot be small. Thus $E \subset H_{ij}$. So E is disjoint from D and contractible, thus an interior (-2)-curve. This is ruled out if all the fibres are generic, so in that case we have only divisorial contractions and the theorem follows as above. Furthermore, in any case, if Z is generic, it is disjoint from the exceptional locus of such a small contraction.

So we need only complete the fibrewise statements (2) and (3) of the theorem. For this we can replace the base by a curve, and assume the base is one-dimensional, and $\mathcal{Z}$ is a smooth three-fold. In this case we can peform the flip. We have seen that the exceptional locus of the small contraction is disjoint from $\mathcal{D}$, and we will prove that the rational map on a fibre Z induced by the flip is a regular isomorphism. Thus the veracity of the statements is not changed by the flip, and we are done by induction. The flip is factored into flips for the small contraction of an irreducible component (these intermediate flips might only be algebraic spaces) which we again call E. Note E is a smooth -2-curve, disjoint from D, contained in a fibre, thus its normal bundle is either $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In either case there is an explicit construction of the flop: a Reid Pagoda flop in the first case, Atiyah's flop in the second. From the explicit expression in [R83] it is clear that Z is unchanged by the flop. This completes the proof of (2) and (3), except for the admissibility of μ if (Z, D) is generic.

However, this argument does show that if Z is generic, then running the $K_{\mathcal{Z}}+\mathcal{D}+\Delta$-MMP will leave a neighborhood of Z unchanged until some class $\mathcal{E}_{ij}|_Z$ is contracted. It then follows by induction that $\{\mathcal{E}_{ij}|_Z\}$ is an exceptional configuration for Z. Since this argument holds for any exceptional configuration on any toric blowup of Y_0, it follows that μ is admissible. $\square$

The following is the key characterization of admissible maps.

Theorem 3.9. (1) *Let E_{ij} and E'_{ij} be two exceptional configurations of the same combinatorial type for generic (Y, D). There is a unique automorphism of* $\operatorname{Pic}(Y)$ *preserving the boundary classes and sending E_{ij} to E'_{ij}. This is admissible. The natural action of* Adm_Y *on the set of exceptional configurations of a given type is simply transitive.*

(2) *Let $(\mathcal{Y}, \mathcal{D}) \to S$ be a family of Looijenga pairs, with S irreducible, and let $g : (Y, D) \to (\mathcal{Y}_v, \mathcal{D}_v)$ and $h : (\mathcal{Y}_w, \mathcal{D}_w) \to (Y, D)$ be isomorphisms identifying (Y, D) with two generic fibres of the family. Pick a path from v to w and let $\mu : \operatorname{Pic}(\mathcal{Y}_w) \to \operatorname{Pic}(\mathcal{Y}_v)$ be the isomorphism given by parallel transport. The automorphism $g^* \circ \mu \circ h^* \in \operatorname{Aut}(\operatorname{Pic}(Y))$ is admissible, and every admissible automorphism is obtained in this way (for some family, some path, and some choice of h, g).*

Remark 3.10. Note that the description of Adm_Y in (2) is equivalent to the usual definition of the monodromy group, i.e., where we require $w = v$ and $h = g^{-1}$. (Indeed, given a family as in the statement, let S' be obtained from S by identifying the points v and w to a point $v' \in S'$, and $(\mathcal{Y}', \mathcal{D}')/S'$ be obtained from $(\mathcal{Y}, \mathcal{D})/S$ by identifying the fiber $(\mathcal{Y}, \mathcal{D})_w$ with $(\mathcal{Y}, \mathcal{D})_v$ by the isomorphism $g \circ h$. Then the path from v to w in S gives a loop in S' based at v' such that, identifying (Y, D) with $(\mathcal{Y}', \mathcal{D}')_{v'} = (\mathcal{Y}, \mathcal{D})_v$ via g, the associated monodromy transformation equals $g^* \circ \mu \circ h^*$.)

Proof. The first statement of (2) follows from Proposition 3.8, as parallel transport gives an admissible isomorphism, and obviously so does any automorphism of a pair. Thus the composition $g^* \circ \mu \circ h^*$ is admissible.

Next let E_{ij} and E'_{ij} be exceptional configurations of the same combinatorial type. As the boundary classes and exceptional classes generate $\mathrm{Pic}(Y)$, there is certainly a unique automorphism of $\mathrm{Pic}(Y)$ preserving the exceptional classes and taking each E_{ij} to E'_{ij}. Clearly any admissible automorphism of $\mathrm{Pic}(Y)$ can be described in this way, for some set of E_{ij}, E'_{ij}. Thus to both show admissibility of this particular automorphism and to show that all admissible automorphisms arise as in (2), it is sufficient to show that this given automorphism of $\mathrm{Pic}(Y)$ can be constructed as in (2).

Passing to the corresponding toric blowup we may assume Y itself has the given toric models. Concretely, we have two expressions for (Y, D) as a blowup of $(\bar{Y}, D)$, one at points $q_{ij} \in D$, the other at points $q'_{ij} \in D$. Now choose a scheme S, points $v, w \in S$, and functions $s_{ij} : S \to D_i^o \subset \bar{Y}$ with $s_{ij}(v) = q_{ij}$, $s_{ij}(w) = q'_{ij}$ (take e.g., S to be a product of copies of D_i^o with s_{ij} the projections) and let $(\mathcal{Y}, \mathcal{D}) \to S \times \bar{Y}$ be the iterated blowup along these sections. Then we have identifications g, h of (Y, D) with the two fibres. Parallel transport in this case is independent of path as exceptional divisors will give a marking of each fibre. The composition clearly sends E_{ij} to E'_{ij}. $\square$

We can now extend the notion of admissible automorphism of $\mathrm{Pic}(Y)$ to the case that (Y, D) is not generic:

Definition 3.11. If (Y, D) is any Looijenga pair and $\tau : (Y, D) \to (Y', D')$ is a toric blow-down, then a *limiting configuration for Y' in* $\mathrm{Pic}(Y)$ *of type $\bar{Y}$ (and τ)* is an ordered collection of classes in $\mathrm{Pic}(Y)$ which are the parallel transport of an exceptional collection for Y'_0 in $\mathrm{Pic}(Y_0)$ of type $\bar{Y}$ (and τ) for a generic deformation $(Y_0, D) \to (Y'_0, D)$ of $(Y, D) \to (Y', D)$.

Let (Y_i, D), $i = 1, 2$ be arbitrary Looijenga pairs. An isomorphism $\mu : \mathrm{Pic}(Y_1) \to \mathrm{Pic}(Y_2)$ is called *admissible* if μ preserves the intersection forms, the classes of the boundary divisors, and gives a bijection on limiting configurations on any toric blowup. As before, $\mathrm{Adm}_Y \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ is the subgroup of admissible automorphisms. Note

that if (Y_i, D) are both generic, then this coincides with the previous definition: by Proposition 3.8, (2), any limiting configuration on a generic (Y, D) is in fact an exceptional configuration.

Corollary 3.12. *Let $(Y_1, D), (Y_2, D)$ be deformation equivalent Looijenga pairs and*

$$\mu : \mathrm{Pic}(Y_1) \to \mathrm{Pic}(Y_2)$$

the isomorphism induced by parallel transport for some path in the base of some smooth family, with some identification of (Y_i, D) with the corresponding fibers. Then μ is admissible, and induces an isomorphism between $\mathrm{Adm}_{Y_1} \subset \mathrm{Aut}(\mathrm{Pic}(Y_1))$ and $\mathrm{Adm}_{Y_2} \subset \mathrm{Aut}(\mathrm{Pic}(Y_2))$.

4. Chambers and universal families

Theorem-Constructon 4.1. *Let (Y_0, D) be a generic Looijenga pair. Let $\{E_{ij}\}, \{F_{ij}\}$ be two exceptional configurations in $\mathrm{Pic}(Y_0)$, not necessarily of the same type. Then there is a canonical birational map*

$$\mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}},$$

commuting with the projections to T_{Y_0}. This birational map has no exceptional divisors and has exceptional locus disjoint from $\mathcal{D}$. Furthermore, it induces an isomorphism when restricted to any given fibre.

Proof. Suppose first the configurations are on Y_0 (rather than on possibly different toric blowups of Y_0). We consider the relative $K_{\mathcal{Y}} + \mathcal{D} + e(\sum \mathcal{F}_{ij})$-MMP on $\mathcal{Y} = \mathcal{Y}_{\{E_{ij}\}}$. By the proof of Proposition 3.8, if we let $T' \subset T_{Y_0}$ be the complement of an appropriate divisor (for example the loci over which any of the $\mathcal{F}_{ij} \to T_{Y_0}$ are non-smooth), then the program applied to the restriction $\mathcal{Y}' := \mathcal{Y} \times_{T_{Y_0}} T'$ is a sequence of divisorial contractions. The end product is a fibre bundle $\overline{\mathcal{Y}} \to T'$ with fibre $\bar{Y}$ the toric variety associated to $\{F_{ij}\}$. Also, $\mathcal{Y}' \to \overline{\mathcal{Y}}$ restricts to an isomorphism of $\mathcal{D} \times_{T_{Y_0}} T'$ with its image in $\overline{\mathcal{Y}}$, and so the markings $p_i : T_{Y_0} \to \mathcal{Y}$ induce markings of $\mathcal{D} \subset \overline{\mathcal{Y}}$, and this, by Proposition 2.5, induces a unique isomorphism $f : ((\overline{\mathcal{Y}}, \mathcal{D}), p_i) \to ((T' \times \bar{Y}, \mathcal{D}), p_i')$ (where p_i' in the second term is given in Construction 3.1 for $\mathcal{Y}_{\{F_{ij}\}}$). Each of $\overline{\mathcal{Y}}$ and $\bar{Y} \times T'$ comes with sections q_{ij} (in the second case by Construction 3.1 for $\mathcal{Y}_{\{F_{ij}\}}$, in the first by the images of the exceptional divisors F_{ij}), which are identified under the isomorphism f. Thus after performing the iterated blow-up of the q_{ij} on $\overline{\mathcal{Y}}$ and $T' \times \bar{Y}$ respectively, f induces an isomorphism $f : \mathcal{Y}' \to \mathcal{Y}_{\{F_{ij}\}}$ onto its image. This yields a birational map $f : \mathcal{Y} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}}$ which is an isomorphism on $\mathcal{D}$.

We need to show this birational map has no exceptional divisors. For this it is enough to restrict the families to a general curve through a given point of the base.

Then we can carry out the MMP. As shown in the proof of Proposition 3.8 the flips are standard K-flops, with exceptional locus an internal -2-curve contained in some F_{ij}. From this the claim is clear.

If the configurations are on toric blowups of Y_0 we make the obvious modifications: we replace Y_0 by a toric blowup $\tau : Z_0 \to Y_0$ on which they both appear and carry out the above. Then we restrict the birational maps to the subtorus $\mathrm{Hom}(\mathrm{Pic}(Y_0), \mathbb{G}_m) \subset \mathrm{Hom}(\mathrm{Pic}(Z_0), \mathbb{G}_m)$, which induce corresponding birational maps between the universal families (which we recall are obtained from these restrictions by a family of toric blowdowns determined by τ). $\square$

Construction 4.2. Observe that $\mathrm{Aut}(\mathrm{Pic}(Y_0))$ acts by precomposition on T_{Y_0}:

$$g(\phi) := \phi \circ g^{-1}.$$

If g is admissible and $\{E_{ij}\}$ is an exceptional collection, then $\{g(E_{ij})\}$ is an exceptional collection necessarily of the same combinatorial type as $\{E_{ij}\}$. This induces, by the construction of the universal families, a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_{\{E_{ij}\}} & \longrightarrow & \mathcal{Y}_{\{g(E_{ij})\}} \\ \downarrow & & \downarrow \\ T_{Y_0} & \xrightarrow{\phi \mapsto \phi \circ g^{-1}} & T_{Y_0} \end{array}$$

where the horizontal maps are isomorphisms. Composing $\mathcal{Y}_{\{E_{ij}\}} \to \mathcal{Y}_{\{g(E_{ij})\}}$ with the canonical birational map $\mathcal{Y}_{\{g(E_{ij})\}} \dashrightarrow \mathcal{Y}_{\{E_{ij}\}}$ gives a birational map

$$\psi_g : \mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{E_{ij}\}},$$

an isomorphism in codimension one. In particular this gives a canonical action of Adm_{Y_0} on $\mathcal{Y}_{\{E_{ij}\}}$ by birational automorphisms which are regular in codimension one. By construction the composition

$$\mathrm{Pic}(Y_0) \xrightarrow{r^{-1}} \mathrm{Pic}(\mathcal{Y}_{\{E_{ij}\}}) \xrightarrow{\psi_{g*}} \mathrm{Pic}(\mathcal{Y}_{\{E_{ij}\}}) \xrightarrow{r} \mathrm{Pic}(Y_0)$$

is $g \in \mathrm{Aut}(\mathrm{Pic}(Y_0))$ (here r is the restriction).

Example 4.3. Consider the pair (Y, D) obtained by blowing up one general point on each coordinate axis of $\mathbb{P}^2$, with D the proper transform of the toric boundary of $\mathbb{P}^2$. Write the generators of $\mathrm{Pic}(Y)$ as L, E_1, E_2, E_3 with L the pull-back of a line in $\mathbb{P}^2$ and the E_i's the exceptional divisors. Then $\{E_1, E_2, E_3\}$ is an exceptional configuration, as is $\{F_1, F_2, F_3\}$ where $F_i = (L - E_1 - E_2 - E_3) + E_i$. We obtain universal families $\mathcal{Y}_{\{E_{ij}\}}, \mathcal{Y}_{\{F_{ij}\}} \to T_Y$. The birational map constructed above $f : \mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}}$ is an isomorphism away from the locus where the three blown-up points lie on a line L'. Over such a point, the curve of class F_i decomposes as a union of irreducible

curves of class $L - E_1 - E_2 - E_3$ and E_k where $\{i, j, k\} = \{1, 2, 3\}$. The curve of class $\alpha := L - E_1 - E_2 - E_3$ is the proper transform of L' and is common to all three curves, hence the three curves cannot be simultaneously contracted. The proper transform of L' must be flopped before this contraction can be performed.

Note in this example that $D^\perp$ is generated by α, and $\Phi = \{\pm\alpha\}$. The reflection s_α satisfies $s_\alpha(E_i) = F_i$. It is an admissible automorphism, and $W = \{\mathrm{id}, s_\alpha\}$. Since the only non-trivial automorphism which preserves the boundary classes and the intersection pairing is s_α, it is clear that $W = \mathrm{Adm}_Y$.

We use the following well-known result:

Lemma 4.4. *Let $\mathcal{Y} \to S$ be a smooth and projective family of surfaces over a smooth curve germ $s \in S$. Let $\mathcal{Y} \to \overline{\mathcal{Y}}$ be the small contraction of a smooth $K_{\mathcal{Y}}$-trivial curve $R \subset Y$, where $Y = \mathcal{Y}_s$ is the fibre. Then $R \subset Y$ is a smooth -2-curve, the flop $m : \mathcal{Y} \dashrightarrow \mathcal{Y}'$ exists, $\mathcal{Y}' \to S$ is again a smooth family, and the induced rational map $Y \to Y'$ is an isomorphism. Moreover the composition*

$$f : \mathrm{Pic}(Y) \xrightarrow{r^{-1}} \mathrm{Pic}(\mathcal{Y}) \xrightarrow{m_*} \mathrm{Pic}(\mathcal{Y}') \xrightarrow{r} \mathrm{Pic}(Y')$$

(where r indicates the restriction of line bundles) is the reflection s_R in the -2-class $R \in \mathrm{Pic}(Y)$.

Proof. For the construction of the flop see [R83]. The result is then an easy computation using Reid's description. $\square$

Corollary 4.5. *Let (Y, D) be a Looijenga pair, and suppose $R \subset Y$ is an internal -2-curve. Then the induced reflection $s_R : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$ is admissible. Thus $W \subset \mathrm{Adm}_Y$.*

Proof. The surface Y occurs as a fibre of a universal family $\mathcal{Y} \to T_Y$. We can restrict the family to a general arc S through $[Y] \in T_Y$, replacing $\mathcal{Y} \to T_Y$ with $\mathcal{Y} \times_{T_Y} S \to S$, so that the curve R can be flopped via a flop $\mathcal{Y} \dashrightarrow \mathcal{Y}'$.

Let $s \in S$ be general, so that $\mathcal{Y}_s = \mathcal{Y}'_s$ canonically, and we then have a composition

$$\mathrm{Pic}(\mathcal{Y}_s) \xrightarrow{r^{-1}} \mathrm{Pic}(\mathcal{Y}) \xrightarrow{m_*} \mathrm{Pic}(\mathcal{Y}') \xrightarrow{r} \mathrm{Pic}(\mathcal{Y}'_s)$$

which is clearly an admissible isomorphism as it is in fact the identity using the canonical identification $\mathcal{Y}_s = \mathcal{Y}'_s$. However, parallel transport using $\mathcal{Y}$ or $\mathcal{Y}'$ gives admissible isomorphisms $\mathrm{Pic}(Y) \cong \mathrm{Pic}(\mathcal{Y}_s)$ and $\mathrm{Pic}(Y) \cong \mathrm{Pic}(\mathcal{Y}'_s)$ by Corollary 3.12, and by Lemma 4.4, the induced admissible automorphism of $\mathrm{Pic}(Y)$ is precisely the reflection s_R. Such reflections by definition generate W, so $W \subset \mathrm{Adm}_Y$. $\square$

Proposition 4.6. *Let $\{E_{ij}\}$ be a limiting configuration in the Picard group of a Looijenga pair (Y,D), associated to a toric model of a toric blowup $Y' \to Y$. Then there is an admissible automorphism g such that $\{g(E_{ij})\}$ is the configuration of exceptional curves for a toric model of Y' of the same type. In particular the existence of a toric model of a given type is deformation invariant.*

Proof. We can assume the toric model is for Y itself. Now let $(\mathcal{Y},\mathcal{D}) \to S$ be a one-parameter family exhibiting $\{E_{ij}\}$ as a limiting configuration, with markings of Pic and D on each fibre. Now we run the $K_{\mathcal{Y}} + \mathcal{D} + \epsilon(\sum \mathcal{E}_{ij})$-MMP relative to S, and induct on the Picard number, as in the proof of Proposition 3.8. A sequence of flops induces an automorphism g of $\operatorname{Pic}(Y)$ consisting of a composition of reflections, by Lemma 4.4. By Corollary 4.5, g is admissible. So, after replacing E_{ij} by $g(E_{ij})$, we can assume the first step in the program is a divisorial contraction, $p : \mathcal{Y} \to \mathcal{Y}_1$, necessarily of some $\mathcal{E}_{ij}$ (which restricts to an internal -1-curve on every fibre). The family $\mathcal{Y}_1$ exhibits $p(\{E_{k\ell} \,|\, (k,\ell) \neq (i,j)\})$ as a limiting configuration on Y_1 (where $p : Y \to Y_1$ blows down E_{ij}). Now by induction there is an admissible automorphism $h : \operatorname{Pic}(Y_1) \to \operatorname{Pic}(Y_1)$ with $h\big(p(\{E_{k\ell} \,|\, (k,\ell) \neq (i,j)\})\big)$ an exceptional configuration. Let $g : \operatorname{Aut}(Y) \to \operatorname{Aut}(Y)$ be the unique automorphism fixing E_{ij} which is compatible with h (i.e., $p_* \circ g = h \circ p_*$). It remains to show g is admissible. But $g(\{\mathcal{E}_{k\ell}\})$ is an exceptional configuration on a general fibre of $\mathcal{Y} \to S$, thus g is admissible by Theorem 3.9. □

Theorem 4.7. *Let $\lambda : ((\mathcal{Z},\mathcal{D}), p_i, \mu) \to S$ be a family of Looijenga pairs deformation equivalent to some generic (Y_0, D), with marked boundaries and Picard groups. Let $s \in S$ be a closed point. There is a Zariski open set $s \in U \subset S$ and an exceptional configuration $\{E_{ij}\}$ in* $\operatorname{Pic}(Y_0)$*, such that after replacing S by U, $((\mathcal{Z},\mathcal{D}), p_i, \mu)$ is isomorphic to the pull-back of some universal family $\mathcal{Y}_{\{E_{ij}\}}$ under the canonical map*

$$\phi_{((\mathcal{Z},\mathcal{D}),p_i,\mu)} : U \to T_{Y_0}$$

of (1.2).

Proof. Passing to a toric blowup (in each fibre of the family) we can assume Y_0 has a toric model, with configuration of exceptional curves $\{F_{ij}\}$. Let (Y,D) be the fibre over s, $\mu : \operatorname{Pic}(Y_0) \to \operatorname{Pic}(Y)$ the marking. By Corollary 4.6, we can find $\{E_{ij}\} = \{g(F_{ij})\}$, for some $g \in \operatorname{Adm}_{Y_0}$, so that $\{\mu(E_{ij})\}$ is the configuration of exceptional curves for a toric model $Y \to \bar{Y}$. It is easy to see this expression will deform to nearby fibres, expressing $(\mathcal{Z},\mathcal{D})|_U$ as an iterated blowup of a fibre bundle with fibre $\bar{Y}$. The pull-back expression follows. □

Lemma 4.8. *Let (Y, D) be a Looijenga pair. Let L be a line bundle on Y such that $L^2 = -2$ and $L|_D \simeq \mathcal{O}_D$. Then $h^0(L) > 0$ or $h^0(L^{-1}) > 0$.*

Proof. Suppose $H^0(L) = 0$. Using the exact sequence

$$0 \to L \otimes \mathcal{O}_Y(-D) \to L \to \mathcal{O}_D \to 0$$

we see that $H^0(L \otimes \mathcal{O}_Y(-D)) = 0$ and $H^1(L \otimes \mathcal{O}_Y(-D)) \neq 0$. Equivalently, by Serre duality, $H^1(L^{-1}) \neq 0$ and $H^2(L^{-1}) = 0$. Now by the Riemann–Roch formula

$$h^0(L^{-1}) > \chi(L^{-1}) = \chi(\mathcal{O}_Y) + \frac{1}{2} L^{-1} \cdot (L^{-1} - K_Y) = 0.$$

□

Definition 4.9. For $\alpha \in \operatorname{Pic}(Y)$ let

$$T_\alpha := \{\phi \in \operatorname{Hom}(\operatorname{Pic}(Y), \mathbb{G}_m) \mid \phi(\alpha) = 1\}.$$

Lemma 4.10. *Let (Y, D) be a Looijenga pair. The following hold:*

(1) *$\phi_Y : D^\perp \to \mathbb{G}_m$ is preserved by W_Y.*
(2) *$\Phi_Y = W_Y \cdot \Delta_Y$.*

Proof. Note $\Delta_Y \subset \Phi_Y \subset \ker(\phi_Y)$ by definitions. Thus if $\alpha \in \Delta_Y$, $s_\alpha \in W_Y$ and $\phi_Y \circ s_\alpha(\beta) = \phi_Y(\beta + \langle \alpha, \beta \rangle \alpha) = \phi_Y(\beta)$. This gives (1).

Now $W \subset \operatorname{Adm}_Y$ by Corollary 4.5. Thus W preserves Φ by Theorem 3.9 (2), and so $W_Y \cdot \Delta_Y \subset \Phi_Y$. To prove the other inclusion, let $\alpha \in \Phi_Y$. By the definition of Φ, there is a family $\mathcal{Z} \to S$ such that (Y, D) is a fibre of this family over $s_1 \in S$ and the class α is realized as an internal (-2)-curve on some other fibre over $s_2 \in S$ of this family. By Theorem 4.7 there are universal families $\mathcal{Y}_i$, $i = 1, 2$ such that in a Zariski neighborhood of s_i, $\mathcal{Z} \to S$ is a pull-back of $\mathcal{Y}_i \to T_Y$, say with fibre $(\mathcal{Y}_1, \mathcal{D})_{x_1} = (Y, D)$ and $(\mathcal{Y}_2, \mathcal{D})_{x_2}$ realizing α as an internal (-2)-curve. Necessarily $x_1, x_2 \in T_\alpha$.

Assume first that $\mathcal{Y}_1 = \mathcal{Y}_2$. Let $\mathcal{Y} \to T_\alpha$ be the base-change of $\mathcal{Y}_1 \to T_Y$ to T_α. By Lemma 4.8, $\alpha \in \operatorname{Pic}(\mathcal{Y})$ is represented by a unique effective divisor $\mathcal{E}$. Replace T_α with a curve $x_1, x_2 \in S$. We run the $K_{\mathcal{Y}} + \mathcal{D} + \epsilon\mathcal{E}$-MMP. As in the proof of Proposition 3.8 after a sequence of flops, which do not change the isomorphism types of the fibres, $\mathcal{E}$ can be contracted by a divisorial contraction of relative Picard number one. So $\mathcal{E} \subset \mathcal{Y}'$ is a smooth $\mathbb{P}^1$-bundle over S. Let $\mathcal{Y} \dashrightarrow \mathcal{Y}'$ be the sequence of flops. By shrinking S, we can assume the exceptional curves of all these flops are contained in the fibre $Y = \mathcal{Y}_{x_1}$. By Lemma 4.4 the composition

$$\operatorname{Pic}(Y) \xrightarrow{r^{-1}} \operatorname{Pic}(\mathcal{Y}) \xrightarrow{m_*} \operatorname{Pic}(\mathcal{Y}') \xrightarrow{r} \operatorname{Pic}(Y)$$

is a composition of reflections by roots in Δ_Y. Clearly $\mathcal{E}|_{Y \subset \mathcal{Y}'} \in \Delta_Y$, thus $\alpha \in W_Y \cdot \Delta_Y$. This shows $\Phi_Y \subset W_Y \cdot \Delta_Y$, and completes the proof of (2) in this case.

Now we consider the possibility that $\mathcal{Y}_1 \neq \mathcal{Y}_2$. By Theorem-Construction 4.1 there is a canonical birational map $\mathcal{Y}_1 \dashrightarrow \mathcal{Y}_2$, commuting with the projections to T_Y, with no exceptional divisors, and with exceptional locus disjoint from $\mathcal{D}$. Further the birational map induced by the restriction to each fibre is an isomorphism. Thus parallel transport $\mathrm{Pic}(Y) \to \mathrm{Pic}((\mathcal{Y}_1)_x)$ and $\mathrm{Pic}(Y) \to \mathrm{Pic}((\mathcal{Y}_2)_x)$ for a general $x \in T_Y$ where the families are the same induces an automorphism of $\mathrm{Pic}(Y)$. To complete the proof it's enough to show that this is a composition of reflections in W_Y. Choose a general curve $x_1 \in C \subset T_Y$, and consider the birational map of three-folds $\mathcal{Y}_1|_C \dashrightarrow \mathcal{Y}_2|_C$. This will be small, with exceptional locus contained in $Y \setminus D$. It follows this is a sequence of flops of internal (-2)-curves, and thus a composition of reflections in W_Y by Lemma 4.4. $\square$

We recall the following statement about the action of Weyl groups:

Theorem 4.11. *The arrangement of hyperplanes*

$$\alpha^{\perp} \subset C^{++}, \quad \alpha \in W_Y \cdot \Delta_Y$$

is locally finite. W_Y acts simply transitively on the Weyl chambers, and each chamber is a fundamental domain for the action of W_Y on C^{++}. One chamber is defined by the inequalities $x \cdot \alpha \geq 0$ for all $\alpha \in \Delta_Y$. The analogous statement holds for the Weyl chambers of C_D^{++}.

Proof. The analogous statement for chambers in C^+ is a basic result in the theory of hyperbolic reflection groups, see [D08], Theorem 2.1. This immediately implies the result for the chambers in C^{++} or C_D^{++} , as these full dimensional subcones of C^+ are preserved by W_Y, see Lemma 2.10. $\square$

Proof of the weak Torelli Theorem. By hypothesis, μ preserves Φ and hence μ preserves $\Phi_Y \subset \Phi$ by condition (4) of the statement of weak Torelli. Also $\Phi_Y = W_Y \cdot \Delta_Y$ by Lemma 4.10. Thus μ permutes the Y-Weyl chambers of C_D^{++}. Since each is a fundamental domain for the action of W_Y on C_D^{++}, there exists a unique $g \in W_Y$ such that $\mu \circ g$ preserves Δ_Y. Now the global Torelli Theorem applies. $\square$

We will now relate the Weyl decomposition of C_D^{++} and the Mori theory of a universal family. So for the next four statements, we fix the following situation. We consider a given universal family $\mathcal{Y} \to T_{Y_0}$, and let $e \in T_{Y_0}$ be the element given by $e(L) = 1$ for all $L \in \mathrm{Pic}(Y_0)$. We will denote by Y_e the fibre of the universal family over e, and denote by Y_g a general fibre which is a generic Looijenga pair.

Lemma 4.12. *Let $\mathcal{Y} \to T_{Y_0}$, Y_e be as above. Then*

$$\Phi = \Phi_{Y_e} = W \cdot \Delta_{Y_e}$$

and

$$W = W_{Y_e}.$$

Proof. Clearly $\Phi = \Phi_{Y_e}$ since ϕ_{Y_e} is trivial. By Lemma 4.10, (2), $\Phi_{Y_e} = W_{Y_e} \cdot \Delta_{Y_e} \subset W \cdot \Delta_{Y_e}$. On the other hand, as in the proof of Lemma 4.10, W preserves Φ, and $\Delta_{Y_e} \subset \Phi$, so $W \cdot \Delta_{Y_e} \subset \Phi$, giving the first displayed formula.

For the second, if α, β are -2-classes, $\gamma = s_\alpha(\beta)$, then $s_\gamma = s_\alpha \circ s_\beta \circ s_\alpha^{-1}$. Since W_{Y_e} is generated by elements of the form s_α, $\alpha \in \Delta_{Y_e}$, it follows that the group generated by elements of $W_{Y_e} \cdot \Delta_{Y_e}$ is precisely W_{Y_e}. On the other hand, $W_{Y_e} \cdot \Delta_{Y_e} = \Phi$ by the first paragraph of the proof, so $W_{Y_e} = W$. □

Lemma 4.13. *Let $\mathcal{Y} \to T_{Y_0}$, Y_e, Y_g be as above. Let $Y \subset \mathcal{Y}$ be any fibre of $\mathcal{Y} \to T_{Y_0}$. We have*

$$\mathrm{NE}(Y_g) \subset \mathrm{NE}(Y) \subset \mathrm{NE}(Y_e)$$
$$\mathrm{Nef}(\mathcal{Y}) = \mathrm{Nef}(Y_e)$$
$$\mathrm{Mov}(\mathcal{Y}) \subset \mathrm{Nef}(Y_g).$$

Proof. We use Lemma 2.9. The cone C^+ and the classes D_i do not vary with the fibres. If $\mathcal{M}(Y)$ denotes the set of irreducible -1-curves in Y not contained in the boundary of Y, then $\mathcal{M}(Y_1) \subset \mathrm{NE}(Y_2)$ for any two fibres Y_1 and Y_2 since any class β with $\beta^2 = K_Y \cdot \beta = -1$ and $H \cdot \beta > 0$ is effective by Riemann-Roch. Furthermore, $\Delta_Y \subset \mathrm{NE}(Y_e)$ by Lemma 4.8 (by considering for $\beta \in \Delta_Y$ the subtorus $T_\beta \subset T_Y$). The first series of inclusions follow since $\Delta_{Y_g} = \emptyset$ by definition of generic. These give the reverse inclusion on nef cones. A moving divisor on $\mathcal{Y}$ will be nef on a general fibre and thus on Y_g, by the inclusions for Nef cones, which is the third inclusion. Clearly $\mathrm{Nef}(\mathcal{Y})$ is the intersection of $\mathrm{Nef}(Y)$ over all fibres, so the second equality follows from the first inclusions. □

Theorem 4.14. *let $\mathcal{Y} \to T_{Y_0}$, Y_e, Y_g be as above.*

(1) *Restriction of line bundles gives identifications* $\overline{\mathrm{Mov}(\mathcal{Y})} = \mathrm{Nef}(Y_g) = \overline{C_D^{++}}$, *and* $\mathrm{Nef}(\mathcal{Y}) = \mathrm{Nef}(Y_e)$. *This identifies the Y_e-Weyl chambers of C_D^{++} with the maximal cones of the Mori fan, which are exactly the cones* $\mathrm{Nef}(\mathcal{Y})$ *over all choices of universal family $\mathcal{Y}$ (for any choice of exceptional configuration on any toric blowup of Y_0).*
(2) Adm_{Y_g} *acts transitively on the Y_e-Weyl chambers, $W \subset$* Adm_{Y_g} *acts simply transitively.*
(3) *The stabilizer in* Adm_{Y_g} *of the chamber* $\mathrm{Nef}(\mathcal{Y})$ *is the image of* $\mathrm{Aut}(Y_e, D)$ *in* Adm_{Y_g}.

Proof. Let $\mathcal{Y} \to T_{Y_0}$ be a given universal family. By Lemmas 2.11 and 4.13, $\mathrm{Nef}(\mathcal{Y}) = \mathrm{Nef}(Y_e)$ is the subcone of $\overline{C_D^{++}}$ cut out by the half-spaces $x \cdot \alpha \geq 0$ for $\alpha \in \Delta_{Y_e}$. On the other hand, $\mathrm{Mov}(\mathcal{Y}) \subset \mathrm{Nef}(Y_g) = \overline{C_D^{++}}$ since Δ_{Y_g} is empty. We have $\Phi = \Phi_{Y_e}$ and $W = W_{Y_e}$ by Lemma 4.12. By Theorem 4.11, $\mathrm{Nef}(Y_e) \cap C^+$ is a fundamental domain for the action of W on C_D^{++}. We also have $W \subset \mathrm{Adm}_{Y_0}$ by Corollary 4.5. By Construction 4.1 the action of Adm_{Y_0} on $\mathrm{Pic}(\mathcal{Y}) = \mathrm{Pic}(Y_0)$ is induced by a birational action (regular in codimension one) on $\mathcal{Y}$. Any birational action regular in codimension one (on any variety) permutes cones of the Mori Fan. In our case, one Weyl chamber, the fundamental domain $\mathrm{Nef}(Y_e) = \mathrm{Nef}(\mathcal{Y})$, is a maximal cone in the Mori fan. Since W acts transitively on the chambers, it follows that every chamber is a maximal cone of the Mori fan. Moreover, since the Mori fan has support contained in $\overline{\mathrm{Mov}(\mathcal{Y})} \subset \mathrm{Nef}(Y_g) = \overline{C_D^{++}}$, we conclude that the closure of the support of the Mori fan equals the closure of the full moving cone, and this in turn is equal to $\mathrm{Nef}(Y_g)$. Thus the intersection of the Mori fan with C_D^{++} is exactly the Weyl chamber decomposition of C_D^{++}. This gives (1-2).

Now consider the stabilizer $S \subset \mathrm{Adm}_{Y_g}$ of $\mathrm{Nef}(\mathcal{Y}) = \mathrm{Nef}(Y_e)$, or equivalently of Δ_{Y_e}. Recall Adm_{Y_g} preserves Φ by (2) of Theorem 3.9. Now S is the image of $\mathrm{Aut}(Y_e, D)$ by the global Torelli Theorem (note the period point ϕ_{Y_e} is trivial so condition (4) of global Torelli is vacuous for $Y = Y_e$). This gives (3). □

Remark 4.15. Friedman shows that the set Φ of roots coincides with the set of classes $\alpha \in \mathrm{Pic}(Y)$ such that $\alpha^2 = -2$, $\alpha \cdot D_i = 0$ for each i, and the associated hyperplane $\alpha^\perp$ meets the interior of the generic nef cone $\mathrm{Nef}(Y_g) = \overline{C_D^{++}}$. See [F12], Theorem 2.14.

Theorem 4.16. *Let $\mathcal{Y} \to T_{Y_0}$, Y_e be as above. Then $W \subset \mathrm{Adm}_{Y_0}$ is a normal subgroup and there is an exact sequence*

$$1 \to \mathrm{Hom}(N', \mathbb{G}_m) \to \mathrm{Aut}(Y_e, D) \to \mathrm{Adm}_{Y_0}/W \to 1$$

where N' is the cokernel of the map

$$\mathrm{Pic}(Y) \to \mathbb{Z}^n, L \mapsto \sum (L \cdot D_i) e_i.$$

Proof. Note $W \subset \mathrm{Adm}_{Y_0}$ by Corollary 4.5. Since, by (2) of Theorem 3.9, Adm_{Y_0} preserves Φ, $W \subset \mathrm{Adm}_{Y_0}$ is normal. The image of $\mathrm{Aut}(Y_e, D)$ in $\mathrm{Aut}(\mathrm{Pic}(Y_0))$ has trivial intersection with $W = W_{Y_e}$, since it preserves the Weyl/Mori chamber $\mathrm{Nef}(Y_e)$, while the Weyl group acts simply transitively on the chambers. Take $g \in \mathrm{Adm}_{Y_0}$. Translating g by an element of W we can assume g preserves the Weyl/Mori chamber $\mathrm{Nef}(Y_e)$, and thus Δ_{Y_e} (as each corresponds to a face of the chamber). Now g is in the image of $\mathrm{Aut}(Y_e, D)$ by the global Torelli Theorem (note the period map is trivial

for Y_e so condition (4) in the theorem is vacuous). Thus $\mathrm{Aut}(Y_e, D) \to \mathrm{Adm}_{Y_0}/W$ is surjective. Now the exactness follows from Proposition 2.4. $\square$

This leads to a simple characterization of admissible automorphisms:

Theorem 4.17. *Let (Y, D) be a Looijenga pair. Then Adm_Y is the subgroup of $\mathrm{Aut}(\mathrm{Pic}(Y))$ preserving the intersection pairing, the boundary classes, and C^{++}.*

Proof. Note Adm_Y satisfies the conditions in the statement by Theorem 3.9. Conversely, let $\mu \in \mathrm{Aut}(\mathrm{Pic}(Y))$ satisfy the conditions of the statement. Note these conditions and Adm_Y are deformation invariant, so we may assume $Y = Y_e$. Note also that μ preserves Φ by Remark 4.15. After translating μ by an element of $W = W_{Y_e} \subset \mathrm{Adm}_Y$, we may assume μ preserves the Weyl/Mori chamber $\mathrm{Nef}(Y_e)$ (see the proof directly above), or equivalently, Δ_{Y_e}. Now μ is induced by an automorphism of (Y_e, D) (and in particular is admissible) by Theorem 1.9. $\square$

Example 4.18. We give an example where Adm_Y/W is nontrivial (in fact, infinite). Let N' be as in the statement of Theorem 4.16 and $K = \mathrm{Hom}(N', \mathbb{G}_m)$. We have an exact sequence

$$0 \to K \to \mathrm{Aut}(Y_e, D_e) \to \mathrm{Adm}_Y/W \to 0.$$

Now let D be a cycle of seven -2-curves. Then one can show that $\mathrm{Aut}(Y_e, D_e)$ is infinite (because there is an elliptic fibration $Y_e \to \mathbb{P}^1$ with infinitely many sections). Moreover N' is finite because $[D_1], \ldots, [D_n] \in \mathrm{Pic}(Y)$ are linearly independent. Hence Adm_Y/W is infinite.

We note by way of comparison:

Lemma 4.19. *In the cases Looijenga considers in* [L81] *we have* $\mathrm{Adm}_Y = W$.

Proof. We use [L81], Proposition I.4.7, p. 284. By definition $\mathrm{Cr}(Y, D)$ is the group of automorphisms of the lattice $\mathrm{Pic}(Y)$ preserving the ample cone of Y and the boundary divisors $D_1, \ldots, D_n$. Let (Y, D) be a very general fiber of the universal family. Then the ample cone of Y is C^{++}, and there are no (-2)-curves on Y, that is, in Looijenga's notation $B^n = \emptyset$. Thus by Theorem 4.17, we have $\mathrm{Adm}_Y \subseteq \mathrm{Cr}(Y, D)$, and [L81], I.4.7 gives $\mathrm{Cr}(Y, D) = W$, the Weyl group. So $\mathrm{Adm}_Y \subseteq W$, but we always have $W \subseteq \mathrm{Adm}_Y$, so $\mathrm{Adm}_Y = W$ as required. $\square$

We now discuss the extent to which we actually have a sensible moduli space of Looijenga pairs. Fix a Looijenga pair (Y_0, D), and let $\mathcal{T}_{Y_0}$ be the functor which assigns to a base S families of Looijenga pairs together with a marking of D, and a marking of Pic by $\mathrm{Pic}(Y_0)$. A universal family does not represent $\mathcal{T}_{Y_0}$: the dependence of the

universal family on the ordering of the blowups implies that $\mathcal{T}_{Y_0}$ is not separated. To get a set-theoretic statement, we can proceed as follows.

The group $\mathrm{Aut}^0(D)\times\mathrm{Adm}_{Y_0}$ acts naturally on $T_{Y_0} = \mathrm{Hom}(\mathrm{Pic}(Y_0), \mathrm{Pic}^0(D))$. Indeed, $\mathrm{Adm}_{Y_0} \subset \mathrm{Aut}(\mathrm{Pic}(Y_0))$ acts on T_{Y_0} by precomposition, while $\mathrm{Aut}^0(D)$ acts via the map $\psi : \mathrm{Aut}^0(D) \to T_{Y_0}$ of Lemma 2.3.

Theorem 4.20. *Let $\lambda : \mathcal{Y} \to T_{Y_0}$ be any universal family, and k an algebraically closed field.*

(1) *There is a natural λ-equivariant action of $\mathrm{Aut}^0(D)$ on $\mathcal{Y}$, and a natural λ-equivariant action of Adm_{Y_0} on $\mathcal{Y}$ by rational automorphisms, commuting with $\mathrm{Aut}^0(D)$, such that for each $g \in \mathrm{Adm}_{Y_0}$, the rational map $g : \mathcal{Y} \dashrightarrow \mathcal{Y}$ is a regular isomorphism generically on every fibre of λ. We set $G = \mathrm{Aut}^0(D) \times \mathrm{Adm}_{Y_0}$.*

(2) *The family λ induces a map*

$$\bar{\lambda} : T_{Y_0}(k) \to \mathcal{T}_{Y_0}(k).$$

The marked period point gives a map in the other direction:

$$\phi : \mathcal{T}_{Y_0}(k) \to T_{Y_0}(k).$$

Then $\phi \circ \bar{\lambda} = \mathrm{id}_{T_{Y_0}(k)}$, and the restrictions of $\bar{\lambda}, \phi$ to the set of generic pairs are inverse maps.

(3) *$\bar{\lambda}$ and ϕ induce isomorphisms*

$$T_{Y_0}(k)/\,\mathrm{Adm}_{Y_0} = \mathcal{T}_{Y_0}(k)/\,\mathrm{Adm}_{Y_0} = \{((Y, D), p_i)\}/\equiv$$
$$T_{Y_0}(k)/G(k) = \mathcal{T}_{Y_0}(k)/G(k) = \{(Y, D)\}/\equiv$$

where in the first instance equivalence is isomorphism of pairs with marking of D, and in the second it is isomorphism of Looijenga pairs.

Proof. For (1), the Adm_{Y_0}-action is described by Construction 4.2. For the $\mathrm{Aut}^0(D)$ action, it is easy to reduce to the case when Y has a toric model, $\pi : Y \to \bar{Y}$. Note that any toric Looijenga pair is generic. Thus by Proposition 3.8, (4), the trivial family

$$\bar{\lambda} : \overline{\mathcal{Y}} := \bar{Y} \times T_{\bar{Y}} \to T_{\bar{Y}},$$

with the sections p_i of Lemma 2.6, (2), and the obvious marking of Pic, finely represents $\mathcal{T}_{\bar{Y}}$. Now $\alpha \in \mathrm{Aut}^0(D)$ can be viewed as acting on $D \times T_{\bar{Y}}$, and consider the family of toric Looijenga pairs $(\overline{\mathcal{Y}}, \alpha\circ p_i) \to T_{\bar{Y}}$. This is isomorphic to a pull-back of the universal family $\bar{\lambda}$ which preserves the sections p_i, by the universal property of $\bar{\lambda}$. One checks the induced action on $T_{\bar{Y}}$ is the given action of $\mathrm{Aut}^0(D)$ on $T_{\bar{Y}}$, using the same argument as in Lemma 2.5. Now consider $T_Y \to T_{\bar{Y}}$, induced by $\pi^* : \mathrm{Pic}(\bar{Y}) \to \mathrm{Pic}(Y)$. This is

$\mathrm{Aut}^0(D)$-equivariant, as follows from the proof of Proposition 2.4, and so the action of $\mathrm{Aut}^0(D)$ on $\overline{\mathcal{Y}} \to T_{\bar{Y}}$ induces an action on the pullback $T_Y \times \bar{Y} \to T_Y$, preserving the p_i. From their definitions, it follows the sections q_{ij} of Construction 3.1 are invariant. Thus the action lifts to the iterated blowup $\mathcal{Y} \to \bar{Y} \times T_Y$. This gives (1).

For (2), $\bar{\lambda}$ exists by definition of the stack $\mathcal{T}_{Y_0}$, the existence of ϕ is obvious, and the last sentence follows from Lemma 3.4 and Proposition 3.8, (4), respectively.

For (3), the second displayed line is clear from the first. The map $\phi : \mathcal{T}_{Y_0}(k) \to T_{Y_0}(k)$ of (2) is obviously Adm_{Y_0}-equivariant, and G-equivariant by Lemma 2.3. It thus induces a map $\phi_Q : \mathcal{T}_{Y_0}(k)/\mathrm{Adm}_{Y_0} \to T_{Y_0}(k)/\mathrm{Adm}_{Y_0}$. The rational action of Adm_{Y_0} on $\mathcal{Y}$ is regular generically on every fibre, and for each $g \in \mathrm{Adm}_{Y_0}$, the restriction to each fibre extends to a regular isomorphism $g : (\mathcal{Y}, \mathcal{D})_x \to (\mathcal{Y}, \mathcal{D})_{g(x)}$. It follows that the composition

$$p \circ \bar{\lambda} : T_{Y_0}(k) \to \mathcal{T}_{Y_0}(k)/\mathrm{Adm}_{Y_0},$$

where $p : \mathcal{T}_{Y_0}(k) \to \mathcal{T}_{Y_0}(k)/\mathrm{Adm}_{Y_0}$ is the quotient, factors through the quotient $T_{Y_0}(k) \to T_{Y_0}(k)/\mathrm{Adm}_{Y_0}$, to give $\bar{\lambda}_Q : T_{Y_0}(k)/\mathrm{Adm}_{Y_0} \to \mathcal{T}_{Y_0}(k)/\mathrm{Adm}_{Y_0}$, a right inverse to ϕ_Q. The map $p \circ \bar{\lambda}$, and thus $\bar{\lambda}_Q$, is surjective by Proposition 3.8, (3). As $\bar{\lambda}_Q$ has a left inverse, it is a bijection. This completes the proof. $\square$

We now consider a special case where we can say more. We say a boundary D of a Looijenga pair is *positive* if the intersection matrix $(D_i \cdot D_j)_{1 \le i,j \le n}$ is not negative semi-definite.

Lemma 4.21. *The following are equivalent for a Looijenga pair (Y, D):*

(1.1) *D is positive.*
(1.2) *There exist integers $a_1, \ldots, a_n$ such that $(\sum a_i D_i)^2 > 0$.*
(1.3) *There exist positive integers $b_1, \ldots, b_n$ such that $(\sum b_i D_i) \cdot D_j > 0$ for all j.*
(1.4) *$U = Y \setminus D$ is the minimal resolution of an affine surface with (at worst) Du Val singularities.*
(1.5) *There exist $1 > c_i > 0$ such that $-(K_Y + \sum c_i D_i)$ is nef and big.*

If any of the equivalent conditions hold, then so do the following:

(2.1) *The Mori cone $\mathrm{NE}(Y)$ is finite rational polyhedral, generated by finitely many classes of rational curves.*
(2.2) $\mathrm{Adm}_Y \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ *is the subgroup preserving the intersection form and the boundary classes, and is a finite group.*
(2.3) *$\Phi \subset D^{\perp}$ is the collection of classes of square -2.*

Proof. (1.1)-(1.5), and (2.1) are contained in [GHKI], Lemma 5.9.

Let $G \subset \mathrm{Aut}(\mathrm{Pic}(Y))$ be the subgroup preserving the intersection pairing and the $[D_i]$. Obviously $\mathrm{Adm}_Y \subset G \subset \mathrm{Aut}(D^{\perp})$. Now $D^{\perp}$ is negative definite by positivity of D and the Hodge index theorem, and thus G is finite. Next we prove (2.3).

By Lemma 4.12, $\Phi = \Phi_{Y_e}$. We can replace Y by Y_e and so may assume that for $\alpha \in \Phi$, $\alpha|_D$ is trivial (as a line bundle). By Lemma 4.10, Φ is preserved by the Weyl group, so we can replace α by $-\alpha$, and so may assume α is effective by Lemma 4.8. Using (1.3) of Lemma 4.21, it follows that α is represented by an effective curve disjoint from D, and thus supported on the exceptional locus of the birational map $Y \to Y'$ given by the nef and big divisor $\sum b_i D_i$ of (1.3), which is the minimal resolution of the surface Y' with Du Val singularities — the exceptional locus is an A, D or E configuration of (-2)-curves. The irreducible components are roots by definition, and any (-2)-class in their span (for example α) is in the Weyl group orbit of these simple roots. This is standard, see e.g., [Bou]. Every (-2)-class is a root by the explicit construction of the A, D, E root systems in [Bou], Chapter VI, §§4.7–4.12, and the Weyl group acts transitively on the set of roots by [Bou], Chapter VI, §1.3, Prop. 11.

Note since D supports a nef and big divisor (which we can take as H in Definition 1.8) G preserves $\mathcal{M}$, C^{++}, and (by (2.3)), the roots Φ. Thus $G = \mathrm{Adm}_Y$ by Theorem 4.17. This proves (2.2). $\square$

Lemma 4.22. *Let (Y, D) be a Looijenga pair. Let $g \in \mathrm{Aut}(Y, D)$ be an automorphism which fixes the boundary divisors pointwise. Then $g^*L - L \in \ker \phi_Y$ for $L \in \mathrm{Pic}(Y)$. If g is non-trivial then ϕ_Y has a non-trivial kernel.*

Proof. The first statement is clear. To establish the second, we must show that if $g^* = \mathrm{id}$ and g fixes the boundary pointwise then $g = \mathrm{id}$. This follows from Proposition 2.4. $\square$

Corollary 4.23. *If (Y, D) is very general in its deformation space, then any automorphism of Y which fixes D pointwise is trivial.*

Proof. For (Y, D) a very general member of any universal family $(\mathcal{Y}, \mathcal{D}) \to T_{Y_0}$, $\phi_Y \in T_{Y_0}$ lies outside the union of all hypertori T_α, $\alpha \in \mathrm{Pic}(Y)$. But if $\alpha \in \ker \phi_Y$, then $\phi_Y \in T_\alpha$. Thus ϕ_Y is injective and the result follows from Lemma 4.22. $\square$

Proposition 4.24. *Let (Y, D) be a Looijenga pair with D positive. Then Y has no non-trivial automorphisms fixing D pointwise.*

We learned the proof of this proposition from R. Friedman.

Proof. Let G be the group of automorphisms of Y fixing D pointwise. The kernel of the homomorphism

$$G \to \mathrm{Aut}(\mathrm{Pic}(Y))$$

is trivial by Proposition 2.4. Also the image is finite because D is positive, see the Proof of Lemma 4.21. Thus G is finite. Let $p \in D$ be a node. Since G is finite, locally analytically near p we can linearize the action. Now the fact that G fixes the two branches of D at p pointwise implies G is trivial. $\square$

We now see in the positive case that T_{Y_0} has a natural functorial meaning. Suppose for $a_i > 0$, $A := \sum a_i D_i$ is ample on Y_0. Let $((\mathcal{Z}, \mathcal{D}), p_i, \mu)$ be any family of marked Looijenga pairs deformation equivalent to (Y_0, D). Then A induces a divisor $\mathcal{A} := \sum a_i \mathcal{D}_i \subset \mathcal{Z}$. Let $((\mathcal{Y}, \mathcal{D}), p_i, \mu) \to T_{Y_0}$ be any universal family. It is easy to show that $\mathcal{A} \subset \mathcal{Y}$ is relatively big and semi-ample. The associated birational morphism $\mathcal{Y} \to \overline{\mathcal{Y}}$ is an isomorphism along $\mathcal{D}$ and has exceptional locus the union of internal (-2)-curves on each fibre. In particular $((\overline{\mathcal{Y}}, \mathcal{D}), p_i)$ is a family of positive pairs with marked boundary, with a simultaneous resolution $(\mathcal{Y}, \mathcal{D}) \to (\overline{\mathcal{Y}}, \mathcal{D})$ together with a marking by $\operatorname{Pic}(Y_0)$. This is universal:

Theorem 4.25. *Let $((\overline{\mathcal{Y}}, \mathcal{D}), p_i)$ be as above. Let $((\overline{\mathcal{Z}}, \mathcal{D}), p_i) \to S$ be a family of positive pairs with marked boundary. Assume there is a simultaneous resolution $(\mathcal{Z}, \mathcal{D}) \to (\overline{\mathcal{Z}}, \mathcal{D})$, with a marking μ by $\operatorname{Pic}(Y_0)$. Then μ induces a canonical period map -*

$$\phi_{((\mathcal{Z},\mathcal{D}),p_i,\mu)} : S \to T_{Y_0}.$$

There is a unique isomorphism of $((\overline{\mathcal{Z}}, \mathcal{D}), p_i)$ with the pull-back under ϕ of $((\overline{\mathcal{Y}}, \mathcal{D}), p_i) \to T_{Y_0}$.

Proof. Let $f : \mathcal{Y}_{\{E_{ij}\}} \dashrightarrow \mathcal{Y}_{\{F_{ij}\}}$ be the rational map between two universal families given by Construction 4.1. Let $\mathcal{A} = \sum a_i \mathcal{D}_i$ be a positive combination (on either). Since f is a regular isomorphism along the boundary, $f^*(\mathcal{A}) = \mathcal{A}$. It follows that the induced map $f : \overline{\mathcal{Y}}_{\{E_{ij}\}} \to \overline{\mathcal{Y}}_{\{F_{ij}\}}$ is an isomorphism. We take $((\overline{\mathcal{Y}}, \mathcal{D}), p_i)$ as $\overline{\mathcal{Y}}_{\{E_{ij}\}}$ for any universal family. The universal property follows easily from Theorem 4.7, and Proposition 4.24. $\square$

Over a Zariski open subset, we obtain a stronger universality:

Theorem 4.26. *Let $((\overline{\mathcal{Y}}, \mathcal{D}), p_i)$ be as above. Let $V_{Y_0} \subset T_{Y_0}$ be the complement of the union $\bigcup_{\alpha \in \Phi} T_\alpha$. Note this is a finite union by Lemma 4.21. Let $\mathcal{V}_{Y_0} \subset \mathcal{T}_{Y_0}$ be the subfunctor of pairs such that D supports an ample divisor. Then the restriction of all universal families to $V_{Y_0} \subset T_{Y_0}$ are the same, and represent $\mathcal{V}_{Y_0}$.*

Proof. We follow the notation of the proof of Theorem 4.25 above. Over $V_{Y_0} \subset T_{Y_0}$, $\mathcal{Y} \to \overline{\mathcal{Y}}$ is an isomorphism, for any universal family, and the fibres are all generic. Now the family $\mathcal{Y} \to V_{Y_0}$ (with its markings of Pic and $\mathcal{D}_i^o$) is universal, by Proposition 3.8. $\square$

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